Nonlinear simulation of trapped, and harmonic, Rossby waves in a channel on the β-plane

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Outline of the talk

1. Theoretical background: An eigenvalue equation for zonally propagating wave solutions of the Shallow Water Equations
2. Harmonic versus Trapped Rossby waves
3. Linear simulations:
   a. The numerical solver
   b. Harmonic waves in a narrow channel
   c. Trapped waves in a wide channel
4. Nonlinear simulations
5. A short summary

Theoretical background: The eigenvalue problem of zonally propagating waves

The vectorial, i.e. coordinate-free, form of the linearized Shallow Water Equation in a rotating system is:

$$\frac{\partial \vec{V}}{\partial t} + f\hat{k} \times \vec{V} = -g\nabla \eta$$

$$\frac{\partial \eta}{\partial t} = -H\nabla \cdot \vec{V}$$

where:

$\vec{V}$ - the 2-Dimensional (horizontal) velocity vector;

f - =2Ωsin(latitude) is the Coriolis frequency (Ω is Earth's rotation frequency);

g - is the gravitational constant (or reduced gravity);

$\hat{k}$ - a unit vector that defines the 2-Dimensional manifold;

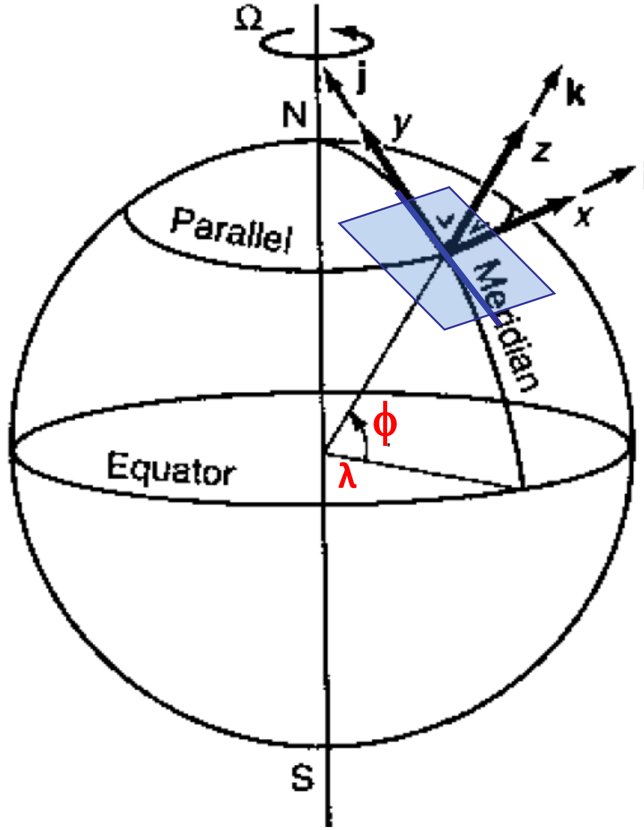
η - the deviation of the free-surface from its mean height, H.

In Cartesian coordinates where *x* points Eastwards and *y* points Northwards the scalar equations are:

$$\frac{\partial u}{\partial t} - (f_0 + \beta y)v = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + (f_0 + \beta y)u = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

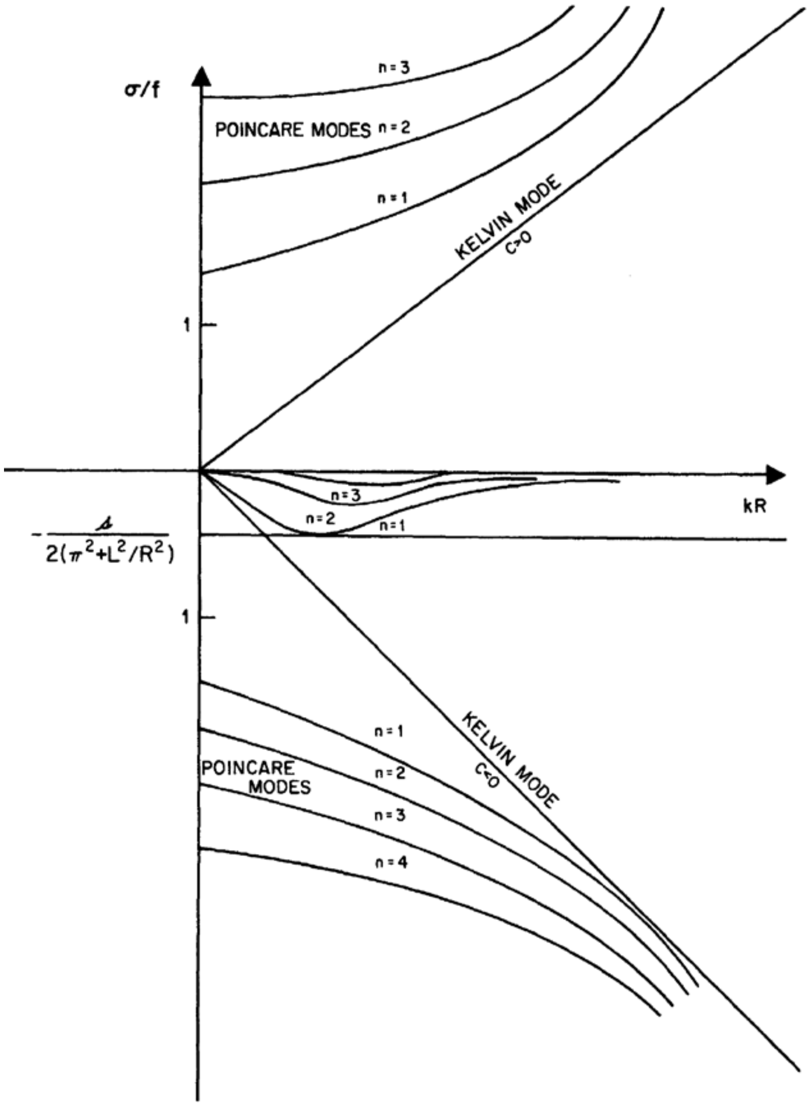


where:

(u, v) are the (East, North) components of $\vec{V}$;

$f_0 + \beta y$ is the Coriolis frequency where: $f_0 = 2\Omega\sin(\phi_0)$ and $\beta = 2\Omega\cos(\phi_0)/a$;

The harmonic, classical, theory of Inertia-Gravity and Planetary waves: (graphs taken from Pedlosky, 1979)



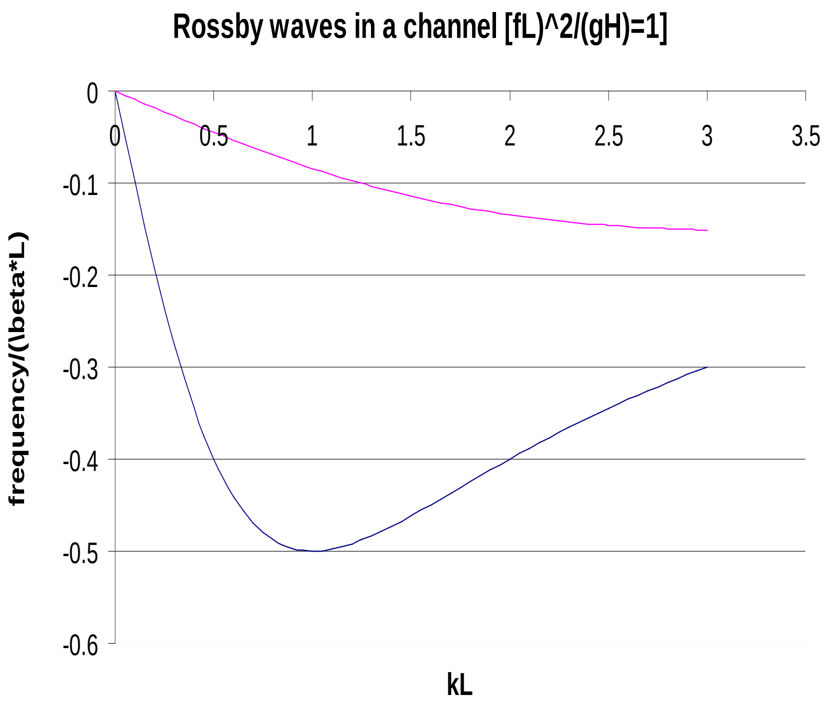
Three types of waves

- Non-dispersive (Kelvin):
  $\sigma = \pm(gH)^{1/2}k$; $v(y) \equiv 0$
  $(gH)^{1/2}$ is the "sound" speed
  (f is in the eigenfunctions)

- Inertia-gravity (Poincare):
  $\sigma = \pm(f^2+(gH)k^2)^{1/2}$

- Planetary (Rossby):
  westward directed phase speed only ($C=\sigma/k<0$; linear with $\beta\equiv df/dy$)

Rossby waves' dispersion on the β-plane in a zonal channel of width L



$$\sigma=-\beta k/(k^2+(m\pi/L)^2+f^2/gH)$$

1. $[m\pi/L, k]$ = [cross-, long-]channel wavenumber ($m = \pm1, \pm2,\ldots$)
2. Propagate westward!
3. Derived from vorticity dynamics

   ⇒"nearly" non-divergent flow

   (⇒"rotational" designation)

   However, non-divergence is essential for time-dependence
4. Although $\beta\neq0$, f_0 is substituted for f ($\equiv f_0+\beta y$) everywhere

A unified eigenvalue equation for all wave types (no large/small spatial scales)

$$\frac{\partial u}{\partial t} - (f_0 + \beta y)v = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + (f_0 + \beta y)u = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial (Hu)}{\partial x} + \frac{\partial (Hv)}{\partial y} = 0$$

- Assume a zonally propagating wave solution: $e^{ik(x-Ct)}$ (C is the phase speed; k is the wavenumber so $kC=\sigma$)
- Eliminate u $\left(= \frac{fV + g\eta}{C}\right)$ (where $V=iv/k$)

$$\frac{dV}{dy} = \left(\frac{f_0 + \beta y}{C} \right) V + \left(\frac{g}{C} - \frac{C}{H} \right) \eta$$

$$\frac{d\eta}{dy} = \left(\frac{k^2 C}{g} - \frac{(f_0 + \beta y)^2}{gC} \right) V - \left(\frac{f_0 + \beta y}{C} \right) \eta$$

$C^2=gH$: (Kelvin waves) the V-equation decouples and its only solution that satisfies 2 BC is $V(y)\equiv 0$

$\Rightarrow$ $\eta(y)$ is exponential with $+y$ for $C>0$ and with $-y$ for $C<0$

$C^2 \neq gH$: The two, 1st order, equations can be transformed into a single, 2nd order, equation for V_{yy}

Neglecting terms proportional to $\beta^2 y^2$ (recall – O(y²) terms were neglected in the expansion of $f(y)$!) yields the 2nd order eigenvalue equation:

$$\frac{d^2V}{dy^2} + \left(E - \frac{2 f_0 \beta}{gH} y \right) V = 0$$

where:

$$E = \frac{\omega^2}{gH} - k^2 - \frac{k\beta}{\omega} - \frac{1}{R_d^2}$$

$$\frac{d^2V}{dy^2}+\left(E-\frac{2f_0\beta}{gH}y\right)V=0$$

Harmonic theory: neglect $\frac{2f_0\beta}{gH}y$

$$\frac{d^2V}{dy^2}+(E)V=0$$

The solutions that satisfy the BC: $V(y=\pm L/2)=0$ (where L is the channel width) are:

$$\sin\left(\frac{n\pi}{L}\left(y+\frac{L}{2}\right)\right)$$

The theory is valid only when $\beta L \ll f_0 \Leftrightarrow L \ll a\cot(\phi_0)$ i.e. when the channel is sufficiently narrow

$$\frac{d^2V}{dy^2} + \left(E - \frac{2f_0\beta}{gH} y \right) V = 0$$

- But, can we do better than that? Can we solve the problem when the neglected O(*y*) term is retained?

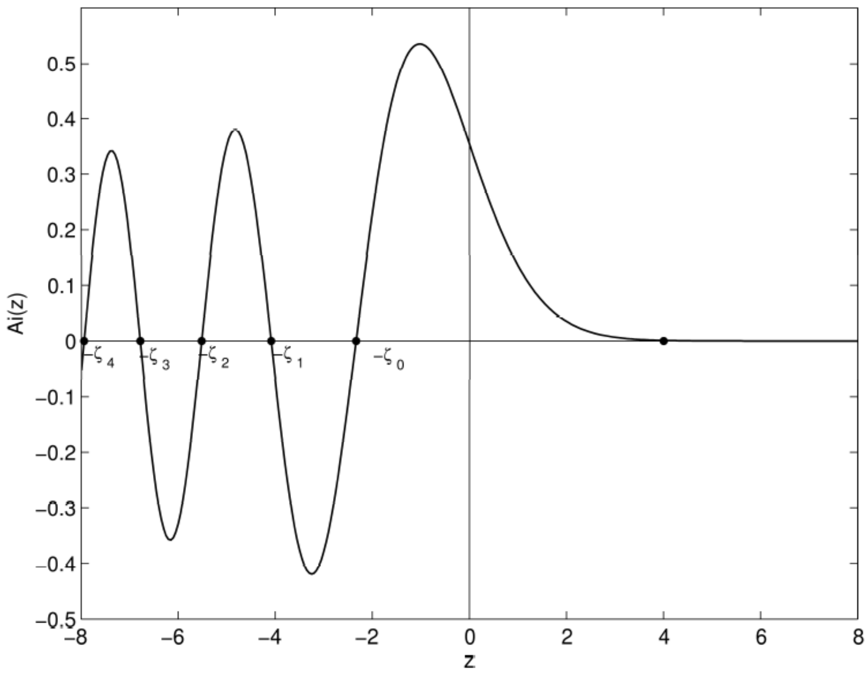
- Is the phase speed of the resulting "non-harmonic" waves higher or lower than that of harmonic waves?

- What is the meridional structure of the amplitudes of these non-harmonic, AKA Trapped, waves

The answers to these questions were given in:
Paldor, Rubin and Mariano (2007, JPO) – Numerical solutions
Paldor and Sigalov (2008, TellusA) – Analytic approximations

A linear change of y to $z(y)$ transforms the equation $\frac{d^2V}{dy^2}+\left(E-\frac{2f_0\beta}{gH}y\right)V=0$ to Airy equation:

$$\frac{d^2V}{dz^2}-zV=0$$



Ai(z): Airy function that vanishes when $z\rightarrow+\infty$ and has isolated zeros at negative z_n ($=-\xi_n$).

[The other solution, Bi(z), is exponentially singular at $z\rightarrow+\infty$ and oscillates at $z<0$]

Summary: Harmonic vs. Trapped waves

Harmonic waves

Meridional amplitude – $v(y)$

$$\sin(n\pi(y+L/2)/L)$$

Dispersion relation

$$\omega = \frac{-\beta k}{k^2 + \frac{1}{R_d^2} + E_n} = \frac{-\beta k}{k^2 + \frac{1}{R_d^2} + \left(\frac{n\pi}{L}\right)^2}$$

Trapped waves

Meridional amplitude – $v(y)$

$$Ai\left(b\left(y + \frac{L}{2}\right) - \zeta_n\right)$$

Dispersion relation

$$\omega = \frac{-\beta k}{k^2 + \frac{1}{R_d^2} + E_n} = \frac{-\beta k}{k^2 + \frac{1}{R_d^2} + b^2\left(\zeta_n - b\frac{L}{2}\right)}$$

where:

$$b = \left(\frac{2 f_0 \beta}{gH}\right)^{1/3} = \left(\frac{2\cot\phi_0}{aR_d^2}\right)^{1/3}$$

The trapped wave theory is valid only in wide channels: $L > \frac{2+\zeta_n}{b}$

Numerical simulation of Harmonic and Trapped Rossby waves in narrow and wide channels

- The (x, y) rectangle domain is a 6,000 km by L km
- Narrow channel: L=500 km; Wide channel: L=3500 km
- The radius of deformation is 531 km (H=200 m)
- A finite difference leap-frog scheme on Arakawa C-grid with uniform resolution of $dx=dy$=10 km
- Periodic BC in x; $k=2\pi/6000\ km^{-1}$
- No-normal flow at channel walls (located at $y=\pm L/2$)
- Initialize the model with Harmonic and Trapped (i.e. Airy) Rossby waves both with n=1
- Both wave types are simulated for 100 days

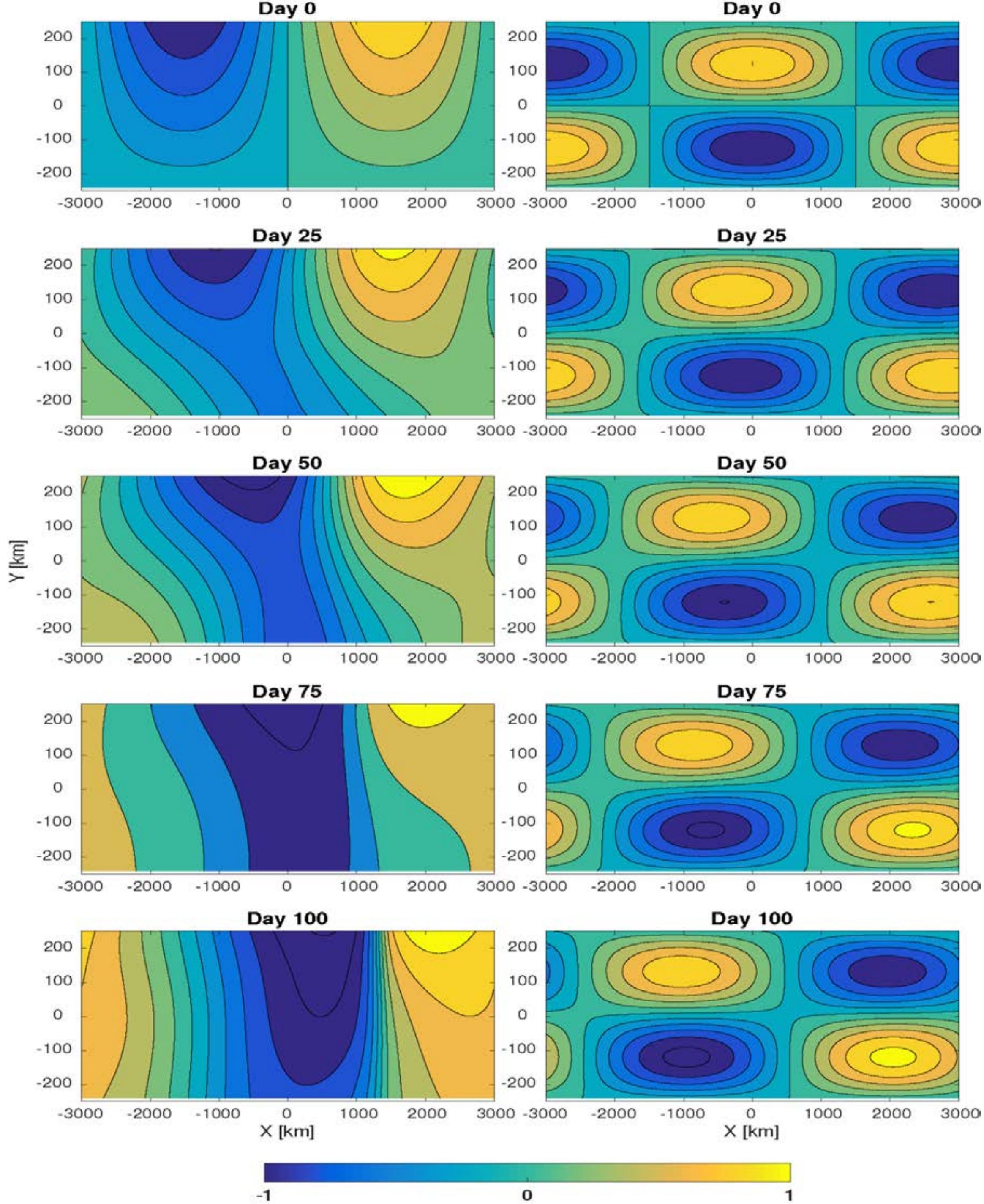
Results (η): Narrow channel

<u>Right</u>: Harmonic

mode propagates at the analytic phase speed of -10km/day

<u>Left</u>: Trapped

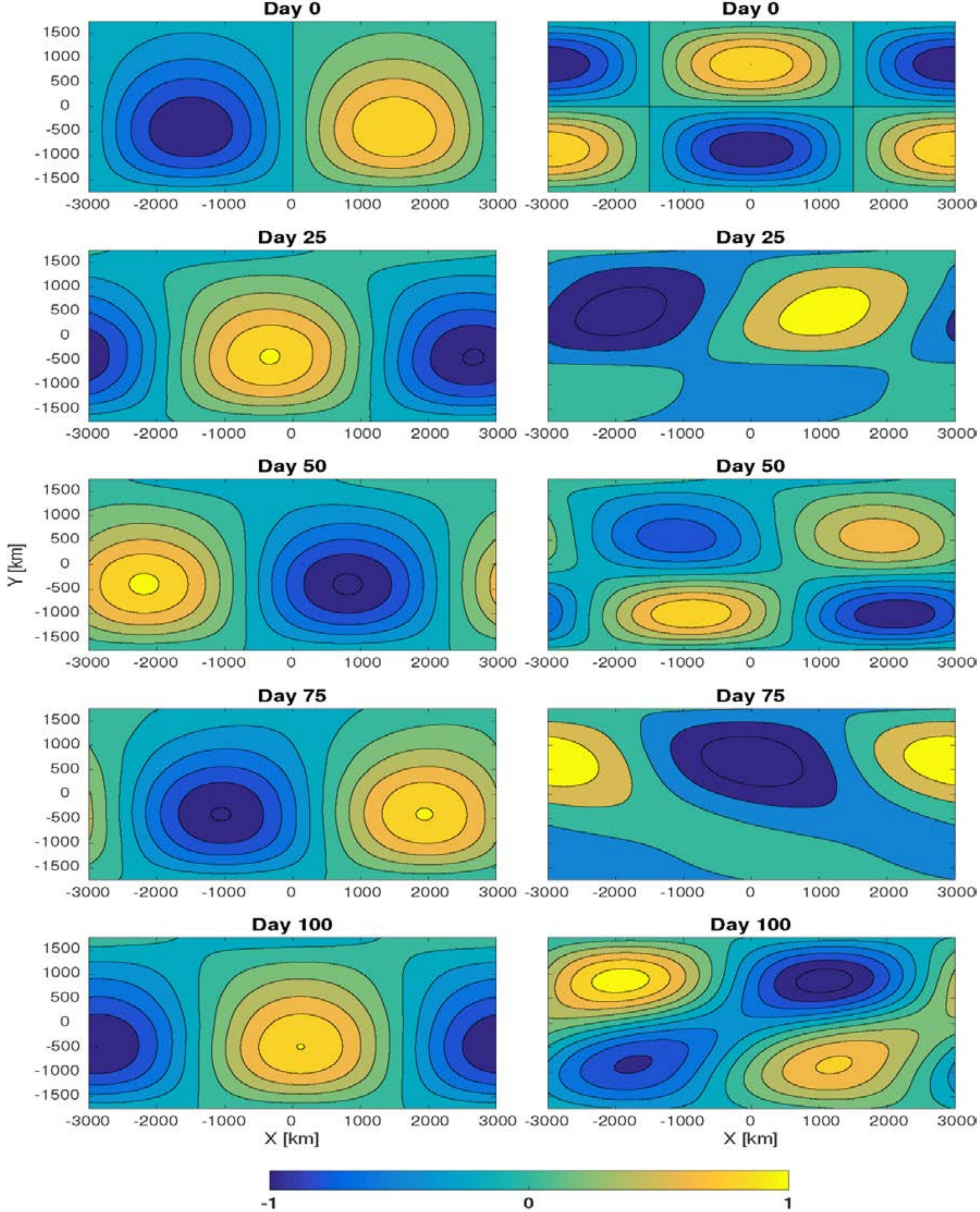
mode deforms



Results (η):
Wide channel:

Right:
Harmonic
mode deforms

Left:
Trapped
mode propagates at the analytic phase speed of -320km/day



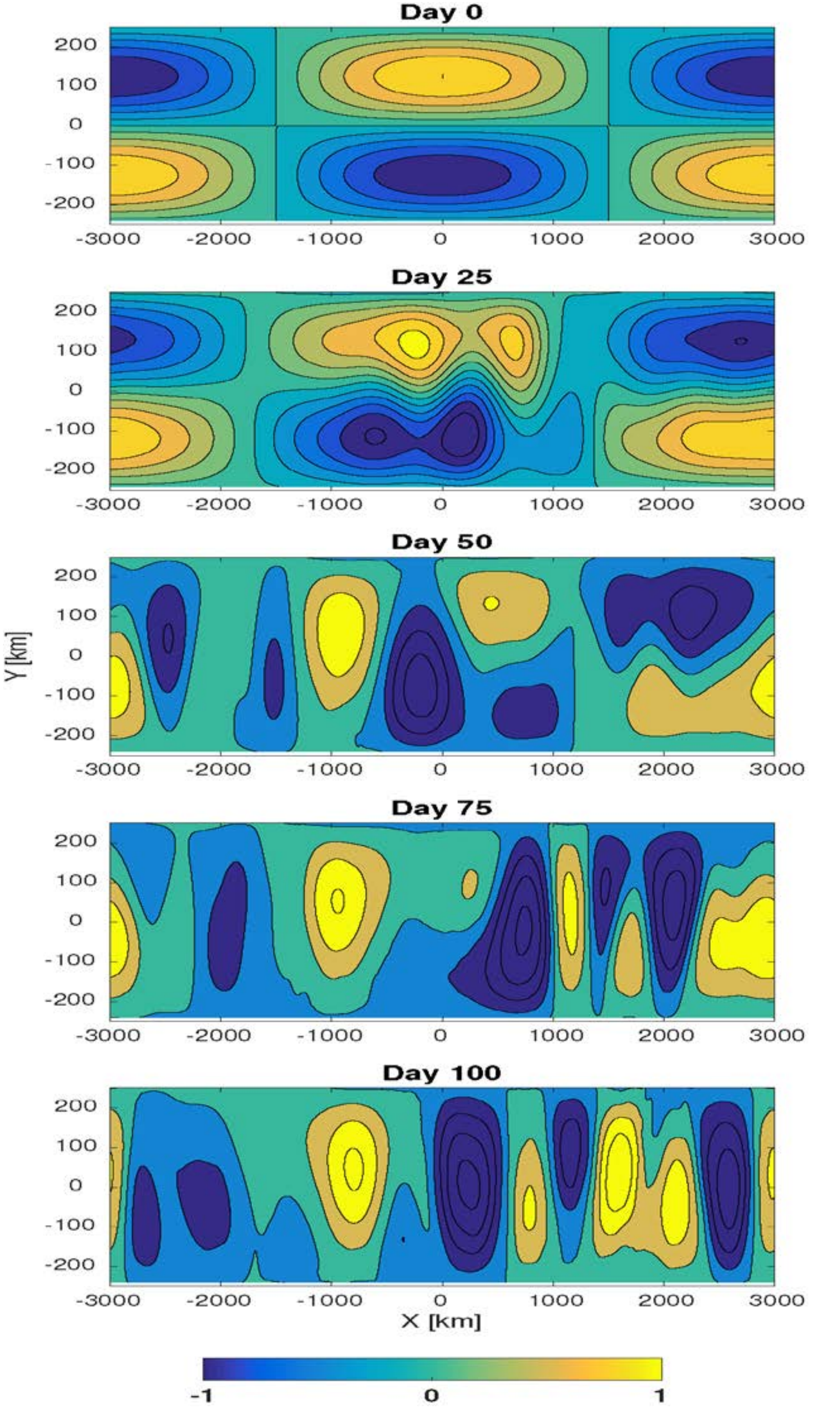
What is the difference between Harmonic, and Trapped, waves in nonlinear simulations?

To answer this question:

- Replace the $\partial/\partial t$ operator by: $D/Dt \equiv \partial/\partial t + V \cdot \nabla$
- Repeat the simulations but now initialize the SW solver with a Trapped mode in a wide channel and with a Harmonic mode in a narrow channel
- Leave all other parameters as in linear simulations:
  - $k=2\pi/6000$ km^{-1}
  - $n=1$
  - $R_d=531$ km;
  - Amplitude of $\eta=1$ m
  - $L=$ 500 km (narrow) and 3500 km (wide)

Nonlinear simulation of a Harmonic mode in a narrow channel:

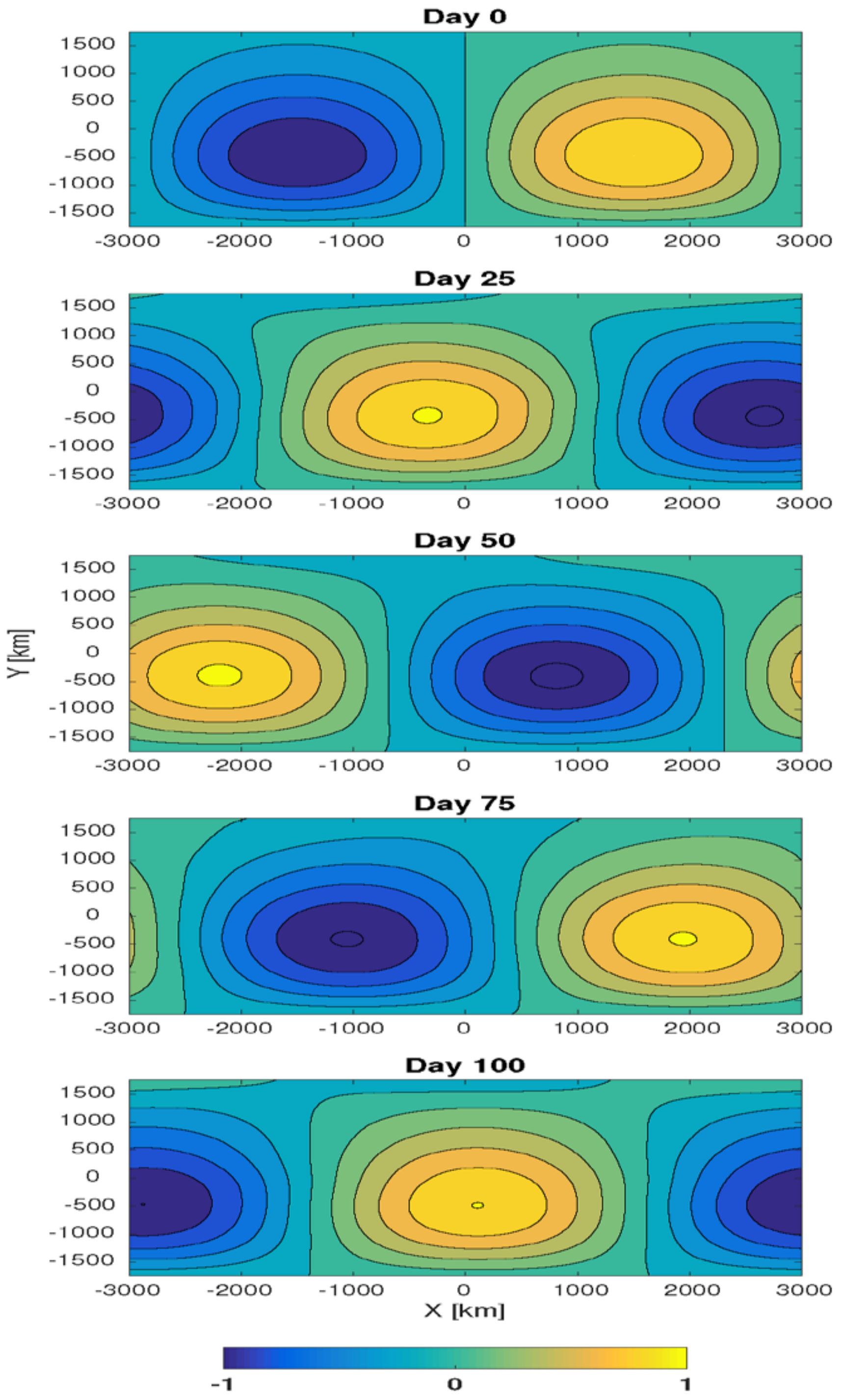
The initial mode is destroyed within O(10-20 days)



Nonlinear simulation of a Trapped mode in a wide channel:

Initial mode is:

1. Preserved for 100 days
2. Propagates at the analytic **LINEAR** phase speed



Summary

1. The analytic wave solutions of the eigenvalue problem provide an accurate description of the linear simulations
2. As in the analytic theory, Harmonic waves approximate the solutions only in narrow channels with width $L << 5.5\left(aR_d^2\right)^{-1/3}$
3. The application of Harmonic waves on the infinite β-plane is unjustified since only Trapped waves approximate the solution in wide channels
4. Need to understand why in nonlinear simulations Trapped waves in a wide channel are preserved for much longer times than Harmonic waves in a narrow channel

Thank you

Gracias