Image Based Analysis of Coherency, Directly From Movies

Erik Boltt, with Abd Al Rahman Al Momani
and with Basaynake

-bolltem@clarkson.edu,  http://www.clarkson.edu/~bolltem
Jupiter Portrait as viewed from Cassini. "This true color mosaic of Jupiter was constructed from images taken by the narrow angle camera onboard NASA’s Cassini spacecraft on December 29, 2000, during its closest approach to the giant planet at a distance of approximately 10 million kilometers (6.2 million miles)."1.
Clearly, time scale and space scale are each important regarding questions concerning some sets “hold together”

Iconic Coherent Set
On certain time scale and space scale, some sets “hold together.”

Many definitions of coherency, and corresponding methods,
A punch-line first – Looking for a “Motion Segmentation” without a vector field
Pondering What to Call Coherence
– Coherence Means A lot of things to A lot of People

- Follow the density? (Ensembles of initial conditions).
  ….flux or transport of density in and out of sets? (Transfer Operator Methods)

- Follow measurements - measurables? (And the Koopman methods)
  (or combination).
- Follow strain of Boundaries? (LCS methods)

- Match shape? Exactly? Approximately? (=> Shape Coherence=> FTC)

Should there be a unique solution to the concept of coherence?
LCS? Coherent Pair? Mesohyperbolic? Shape Coherent? Exact Coherent?

- A clustering method

- Follow the “surprise” – information theory story

ALL of these need a step to develop a model as SOME WAY to follow particles – a flow map, a vector field, etc.

What can we do DIRECTLY from watching a movie?
From my previous discussion - Sets that keep their shape “look” coherent.

The Green Set is Such a Set

Tian Ma and Erik M. Bollt, "Differential Geometry Perspective of Shape Coherence and Curvature Evolution by Finite-Time Nonhyperbolic Splitting, (SIADS), 2014
Coherence vs continuity
– image of a connected set is connected

Before

After

The Green Set is Such a Set

-Froyland, Junge
-G. Haller & F.J. Beron-Vera
Modelling the vector field of the flow from images

-again - can we skip this step?

Flow from the Quasi-Geostrophic Multi-time step method with n=2

Basaynake, Bollt
In an Effort as SOME WAY to follow particles

\[ \dot{x} = F(x, t), \]

\[ F : M \rightarrow M \]

a flow mapping, \( x(t) = \Phi(x_0, t_0, t) \) is inferred
But do we really need the vector field?
A hyperbolic coherent set vs an elliptic coherent set

All the particles move together BUT they mix/shuffle within the coherent

Frobenius-Perron – $P$ - advance density,
Koopman - $U$ - evaluate a function at the image,
$U \circ P[\rho_0](x) = U[\rho_1](x) = \rho_1 \circ T(x)$ evaluate the new density at the image point,
-or pull back density and look for almost invariant sets
Can Coherence be Found without a Vector Field? Just based on what we watch?

Goal – coherent when we see it, and to develop appropriate mathematical formalism behind this sense.
A punch-line first – Looking for a “Motion Segmentation” without a vector field
From The parallel worlds of spectral partitioning, graph theory, data analytics, image processing and dynamical systems coherency


There have been several complementary views of clustering by spectral methods, by graph cuts as random walkers, and comparably as a diffusion process as described by diffusion map and comparably as an eigensystem.

**Weighted Directed Spectral Graph Theory** has special issues related to the conductance of a graph (often called Cheeger-constant) which is numerical measure of whether or not a graph has a "bottleneck", and it given by:

\[
\phi(G) = \min_{S \subseteq V} \phi(S)
\]

Where \( \phi(S) = \frac{\sum_{i \in S, j \in \bar{S}} a_{ij}}{\min(a(S), a(\bar{S}))} \) is the conductance of the cut \((S, \bar{S})\) of the graph \(G\), and \(a_{ij}\) are entries of adjacency matrix of \(G\).

\[
X = \begin{bmatrix} C_1^T \\ C_2^T \\ \vdots \\ C_p^T \\ \vdots \\ C_{pq}^T \end{bmatrix}
\]

So for color alone, \(X\) is \(d \times pq\), and write \(X_i = C_i^T\), the column vector of colors at pixel position \(i\). Then the pairwise distance is:

\[
D_{i,j} = \|X_i - X_j\|_2
\]

Which gives the pairwise symmetric affinity matrix:

\[
W_{i,j} = e^{-D_{i,j}^2/2\sigma^2}.
\]

Degree matrix:

\[
D_{i,i} = \sum_j W_{i,j}, \quad D_{i,j} = 0, \quad i \neq j
\]

Then we have maximum cut problem:

\[
\min_{x} \text{ncut}(x) = \min_y \frac{y^T(D - W)y}{y^T Dy}
\]

Which solved by the generalized eigenvalue eigenvector problem

\[
(D - W)y = \lambda Dy
\]

Bottleneckyness
Weighted Directed Affinity For Developing “Motion Segmentation”
interpretation of coherence without a vector field

\[ X = \left[ C_{1,:,:}(t)^T | C_{2,:,:}(t)^T | \ldots | C_{pq,:,:}(t)^T \right] \]

\[ D_1(i, j, a, \tau) = \sum_{l=1}^{T-1} \| X_i(t + la) - X_j(t + (l - 1)a) \|_2, \text{ Measure Affinity} \]
\[ D_2(i, j)^2 = \| z(i) - z(j) \|_2^2, \text{ Spatial Displacement Affinity} \]

\[ D(i, k, a, \tau)^2 = D_1(i, j, a, \tau)^2 + \alpha D_2(i, j)^2 \]

The Weighted Directed Affinity
\[ \mathcal{W}_{i,j} = e^{-D(i,k,a,\tau)^2/2\sigma^2} \]

\[ \mathcal{P} = \mathcal{D}^{-1} \mathcal{W} \]

Where \( \mathcal{D} \) is the degree matrix and \( \mathcal{P} \) is a row stochastic matrix representing probabilities of a Markov chain through the directed graph.

Note: Affinity is based only in “affinity” from movie, no trajectories or transfer operators.
Directed Graph Analogue of Graph Laplacian

\[ \mathcal{L} = I - \frac{\Pi^{1/2} P \Pi^{-1/2} + \Pi^{-1/2} P^T \Pi^{1/2}}{2} \]

Where \( \Pi \) is the diagonal matrix from the eigenvalue problem \( u = u P \), and \( \Pi = \text{diag}(u) \).

**Numerical Results**

Courant-Fischer Formula and Raleigh Quotient

Then eigenvectors cluster, Max-cut problem for directed – bottleneckyness/Cheeger/ conductance

From Chung, -already standard FP operators


\[ \lambda_2 = \min_{x^T p^{1/2} = 0, x \neq 0} \frac{x^T \mathcal{L} x}{x^T x} = \min_{y^T p = 0, y \neq 0} \frac{\sum_{i,j} (y_i - y_j)^2 p_i P_{i,j}}{\sum_i y_i^2 p_i} \]
Zonal Flow The wind speed in Jupiter’s atmosphere, measured relative to the planet’s internal rotation rate. Alternations in wind direction are associated with the atmospheric band structure.
What is this?

• Coherent Sets?
• Clustered Time Evolving Measurements?
• Advection only?
Does observed coherence correspond to the same coherence one would get with a transfer operator from an advective process?

Weather - reaction (diffusive) advective, vs purely advective

*Cupid* Note to Self:
Never Fall Asleep in a Blizzard
Weather - advective?
And all of our **favorite** benchmark test
In the subject of coherency

The “Harmonic Oscillator” of coherency

The **Double Gyre**
The End

-If you have the vector field, us it
-If you have the transfer operator, us it
-Coherence as a spectral motion tracking

-Modified with a weighted directed graph Laplacian – background of Cheeger by Fischer-Courant and Raleigh Quotient

-Details in papers on my website, and on the weighted directed graph Laplacian in my book
-Relate to an advective diffusion map