Parametric instability of a many point-vortex system in a multi-layer flow under linear deformation

E. A. RYZHOV, K. V. KOSHEL

PACIFIC OCEANOLOGICAL INSTITUTE, FEB RAS,

43 BALTIYSKAYA STREET, VLADIVOSTOK, 690041, RUSSIA

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I. General results for absolute and relative motion of point-vortex systems in a multi-layer flow under linear deformation

We consider a dynamical system governing the motion of many point vortices located in the different layers of a multi-layer quasi-geostrophic flow under external deformation.

$$\psi = \psi_b + \psi_v, \quad \psi_b = S(t)\left(\left(x - x_0\right)^2 - \left(y - y_0\right)^2\right) + \Omega(t)\left(\left(x - x_0\right)^2 - \left(y - y_0\right)^2\right),$$

For method of calculating ψ_v in multilayer model see

Gryanik, V. M. and Tevs, M. V. (1989) Dynamics of singular geostrophical vortices in a N-lavel model of the atmosphere (ocean) Izvestiya atmospheric and oceanic physics **25** 179-188.

$$\frac{dx_j^\alpha}{dt} = A(t) + 2y_j^\alpha\left(S(t) - \Omega(t)\right) - \sum_{i=1}^{M} h_i \sum_{\beta=1}^{N_i} \mu_i^\beta \frac{(y_j^\alpha - y_i^\beta)}{\left(r_{ji}^{\alpha\beta}\right)^2} \Phi_{ji}^{\alpha\beta}\left(r_{ji}^{\alpha\beta}\right),$$

$$\frac{dy_j^\alpha}{dt} = -B(t) + 2x_j^\alpha\left(S(t) + \Omega(t)\right) + \sum_{i=1}^{M} h_i \sum_{\beta=1}^{N_i} \mu_i^\beta \frac{(x_j^\alpha - x_i^\beta)}{\left(r_{ji}^{\alpha\beta}\right)^2} \Phi_{ji}^{\alpha\beta}\left(r_{ji}^{\alpha\beta}\right). \tag{1}$$

$$\Phi_{ji}^{\alpha\beta}\left(r_{ji}^{\alpha\beta}\right) = a_{ji}^{\alpha\beta 0} + a_{ji}^{\alpha\beta 1} r_{ji}^{\alpha\beta} K_1\left(b_{ji}^{\alpha\beta 1} r_{ji}^{\alpha\beta}\right) + a_{ji}^{\alpha\beta 2} r_{ji}^{\alpha\beta} K_1\left(b_{ji}^{\alpha\beta 2} r_{ji}^{\alpha\beta}\right).$$

$$\mu = \sum_{i=1}^{M} \sum_{\beta=1}^{N_i} h_i \mu_i^\beta \neq 0,$$

$$X = \frac{1}{\mu} \sum_{i=1}^{M} \sum_{\beta=1}^{N_i} h_i \mu_i^\beta x_i^\beta, \quad Y = \frac{1}{\mu} \sum_{i=1}^{M} \sum_{\beta=1}^{N_i} h_i \mu_i^\beta y_i^\beta.$$

$$
\begin{aligned}
\frac{dX}{dt} &= A(t) + 2Y\left(S(t) - \Omega(t)\right), \\
\frac{dY}{dt} &= -B(t) + 2X\left(S(t) + \Omega(t)\right).
\end{aligned}
\qquad (2)
$$

$$
\begin{pmatrix} X \\ Y \end{pmatrix} = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} + \mathbf{X}(t)\int_{t_0}^{t}\mathbf{X}^{-1}(\tau)\begin{pmatrix} A(t) \\ -B(t) \end{pmatrix}d\tau
$$

$$
\begin{aligned}
\frac{dX}{dt} &= 2Y\left(S(t) - \Omega(t)\right), \\
\frac{dY}{dt} &= 2X\left(S(t) + \Omega(t)\right).
\end{aligned}
\qquad (3)
$$

$$Z_1 = X / \sqrt{\left(S(t) - \Omega(t)\right)},\ Z_2 = Y / \sqrt{\left(S(t) + \Omega(t)\right)}$$

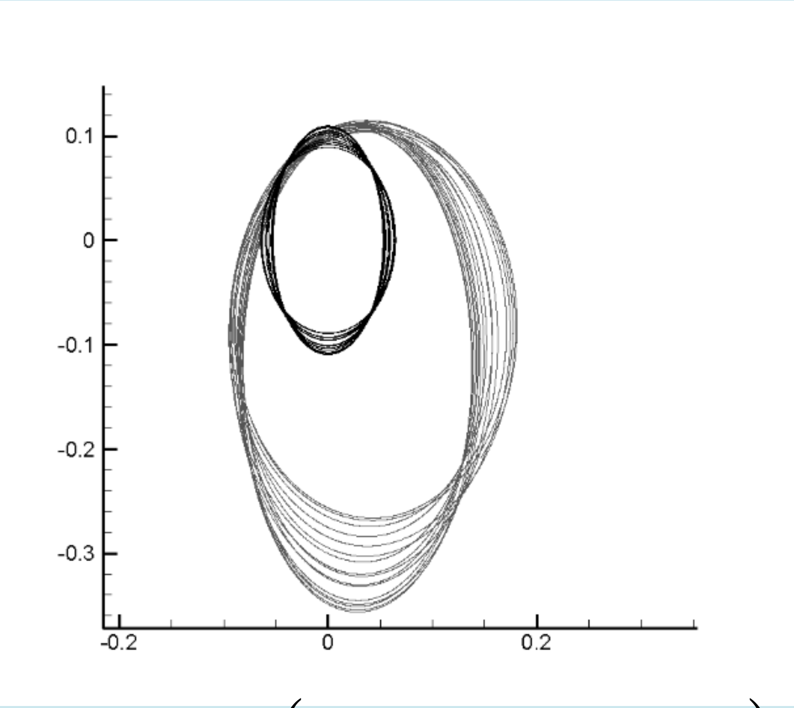
$$\frac{d^2 Z_{1,2}}{dt^2} + \left\{ \frac{1}{2} \frac{d^2 \ln\left(S(t) \pm \Omega(t)\right)}{dt^2} - \frac{1}{4}\left[\frac{d \ln\left(S(t) \pm \Omega(t)\right)}{dt} \right]^2 - 4\left(S^2(t) - \Omega^2(t)\right) \right\} Z_{1,2} = 0,$$

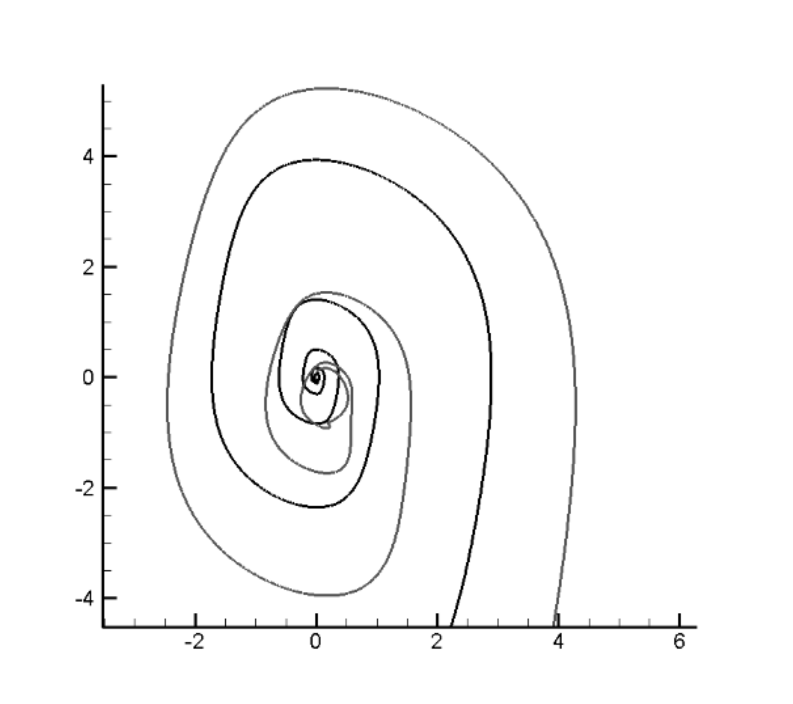
$S(t) = S_0 \Phi_1(t);\ \Omega(t) = \Omega_0 \Phi_2(t)$ in case of $\Phi_1(t) = \Phi_2(t) = \Phi(t)$

$$X = X_0 \cos\left(2\sqrt{\left(\Omega_0{}^2 - S_0{}^2\right)} \int_0^t \Phi(\tau) d\tau \right) - \sqrt{\frac{\left(\Omega_0 - S_0\right)}{\left(S_0 + \Omega_0\right)}} Y_0 \sin\left(2\sqrt{\left(\Omega_0{}^2 - S_0{}^2\right)} \int_0^t \Phi(\tau) d\tau \right),$$

$$Y = Y_0 \cos\left(2\sqrt{\left(\Omega_0{}^2 - S_0{}^2\right)} \int_0^t \Phi(\tau) d\tau \right) + \sqrt{\frac{\left(S_0 + \Omega_0\right)}{\left(\Omega_0 - S_0\right)}} X_0 \sin\left(2\sqrt{\left(\Omega_0{}^2 - S_0{}^2\right)} \int_0^t \Phi(\tau) d\tau \right).$$

V. Rom-Kedar, A. Leonard, S. Wiggins, J. Fluid Mech. 214 (1990) 347.
Koshel K. V. and Ryzhov E. A., Phys. Lett. A, 376 (2012) 744.





$$S = -0.01(1 + 0.5\sin 0.01t);$$

$$\Omega = -0.02(1 + 0.2\sin 0.01t).$$

$$S = -0.01(1 + 0.5\sin 0.07t);\ \Omega = -0.02.$$

Vorticity center trajectories in the bounded motion regime (left), and unbounded motion regime (parametric instability case) (right). The darker trajectories correspond to the absence of the uniform flow A = B = 0, the grey trajectories correspond to $A = 0.01(1 + 0.1\sin 0.01t);\ B = 0.01(1 + 0.1\sin 0.01t).$

$$\tilde{x}_j^\alpha = x_j^\alpha - X;\ \tilde{y}_j^\alpha = y_j^\alpha - Y,$$

$$\frac{d\tilde{x}_j^\alpha}{dt} = 2\tilde{y}_j^\alpha\left(S(t) - \Omega(t)\right) - \sum_{i=1}^{M} h_i \sum_{\beta=1}^{N_i} \mu_i^\beta \frac{(\tilde{y}_j^\alpha - \tilde{y}_i^\beta)}{\left(\tilde{r}_{ji}^{\alpha\beta}\right)^2} \Phi_{ji}^{\alpha\beta}\left(\tilde{r}_{ji}^{\alpha\beta}\right),$$

$$\frac{d\tilde{y}_j^\alpha}{dt} = 2\tilde{x}_j^\alpha\left(S(t) + \Omega(t)\right) + \sum_{i=1}^{M} h_i \sum_{\beta=1}^{N_i} \mu_i^\beta \frac{(\tilde{x}_j^\alpha - \tilde{x}_i^\beta)}{\left(\tilde{r}_{ji}^{\alpha\beta}\right)^2} \Phi_{ji}^{\alpha\beta}\left(\tilde{r}_{ji}^{\alpha\beta}\right), \qquad (1')$$

We have two integrals for relative vortex motion: $\tilde{X} = \tilde{Y} = 0$

Many results for that case $X = Y = 0$ (see, for example M. A. Sokolovskiy, J. Verron, Dynamics of vortex structures in a stratified rotating fluid, Springer, 2014.) can be spreaded to the case of the arbitrary motion of vorticity center.

II. Simple example:
Two vortex with arbitrary intensities in two layer model.

$$\frac{d\tilde{x}_j}{dt} = 2\tilde{y}_j\left(S(t)-\Omega(t)\right) - h_{3-j}\mu_{3-j}\frac{(\tilde{y}_j-\tilde{y}_{3-j})}{\left(\tilde{r}_{j(3-j)}\right)^2}\left[1-\gamma\tilde{r}_{j(3-j)}K_1\left(\gamma\tilde{r}_{j(3-j)}\right)\right],$$

$$\frac{d\tilde{y}_j}{dt} = 2\tilde{x}_j\left(S(t)+\Omega(t)\right) + h_{3-j}\mu_{3-j}\frac{(\tilde{x}_j-\tilde{x}_{3-j})}{\left(\tilde{r}_{j(3-j)}\right)^2}\left[1-\gamma\tilde{r}_{j(3-j)}K_1\left(\gamma\tilde{r}_{j(3-j)}\right)\right],$$

$$h_1m_1\tilde{x}_1 + h_2m_2\tilde{x}_2 = 0,\ \ h_1m_1\tilde{y}_1 + h_2m_2\tilde{y}_2 = 0,$$

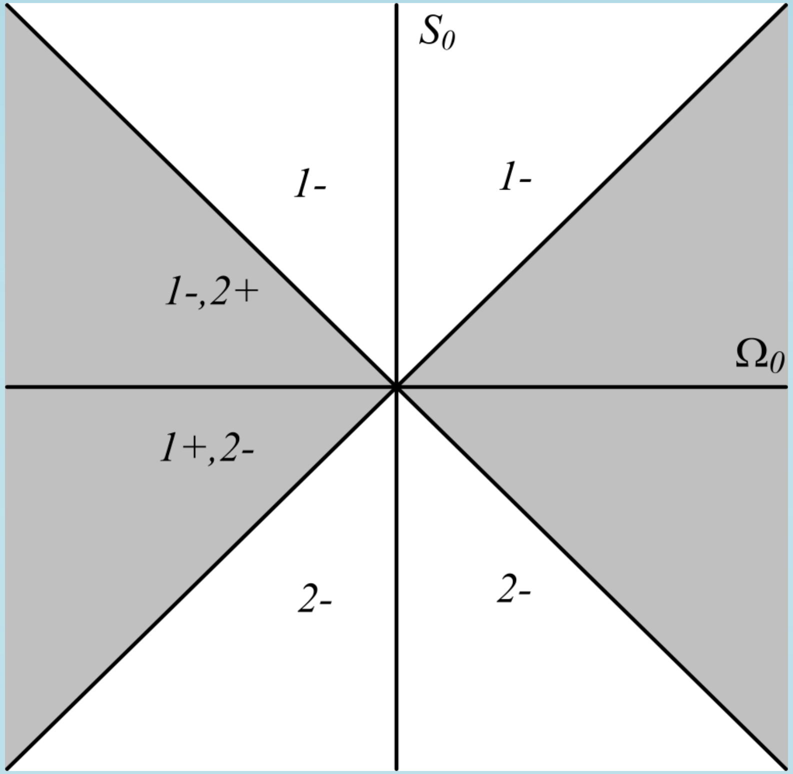
$$\tilde{x}_1 - \tilde{x}_2 = \tilde{x}_1\left(1+\frac{h_1m_1}{h_2m_2}\right),\ \tilde{y}_1 - \tilde{y}_2 = \tilde{y}_1\left(1+\frac{h_1m_1}{h_2m_2}\right),\ \tilde{r}_{12} = \tilde{r}_1\left|1+\frac{h_1m_1}{h_2m_2}\right|.$$

$$\frac{dx_1}{dt} = y_1\left(2\left(S(t)-\Omega(t)\right)-\frac{\mu}{r_1}\left[\frac{1}{r_1}-K_1(r_1)\right]\right),$$

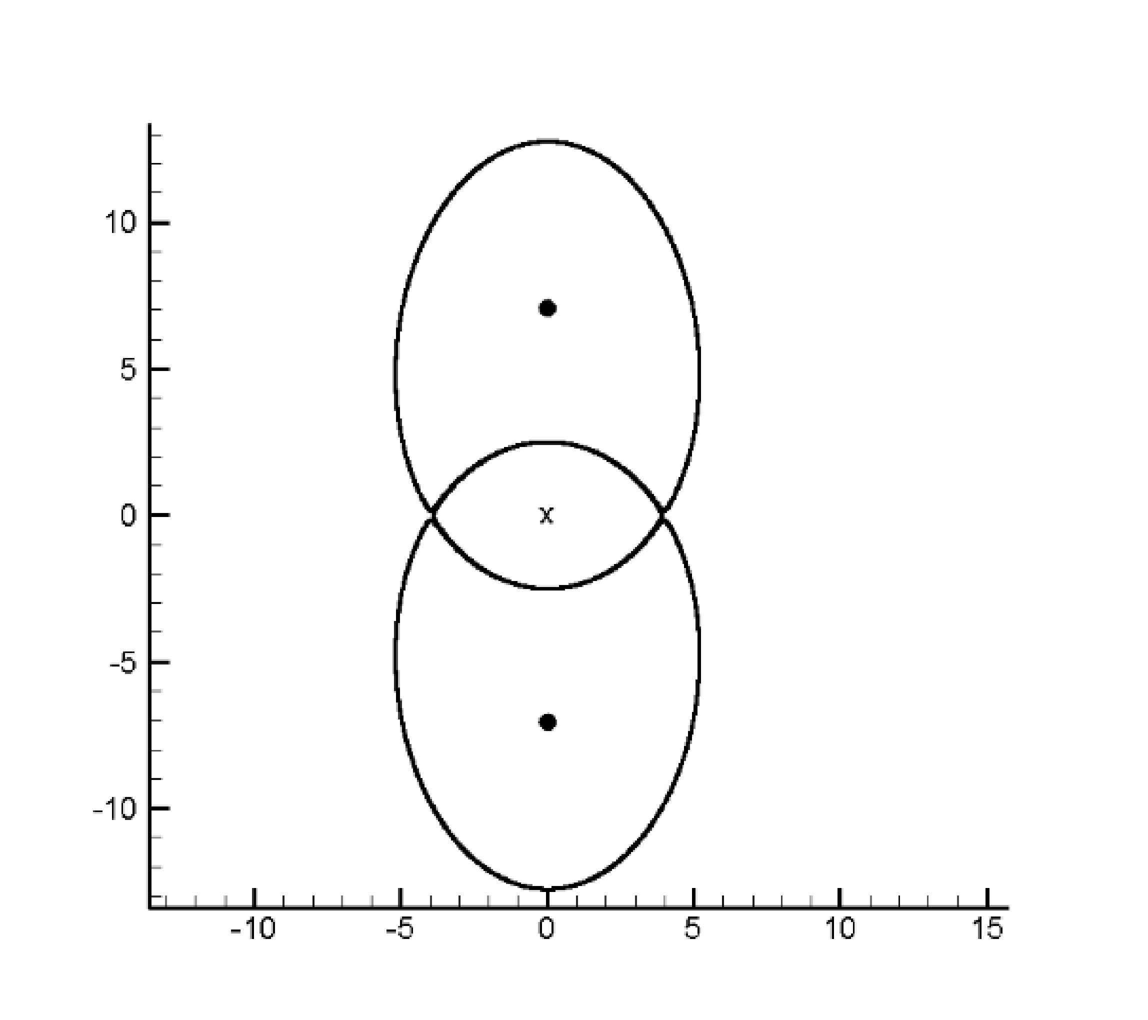
$$\frac{dy_1}{dt} = x_1\left(2\left(S(t)+\Omega(t)\right)+\frac{\mu}{r_1}\left[\frac{1}{r_1}-K_1(r_1)\right]\right).$$

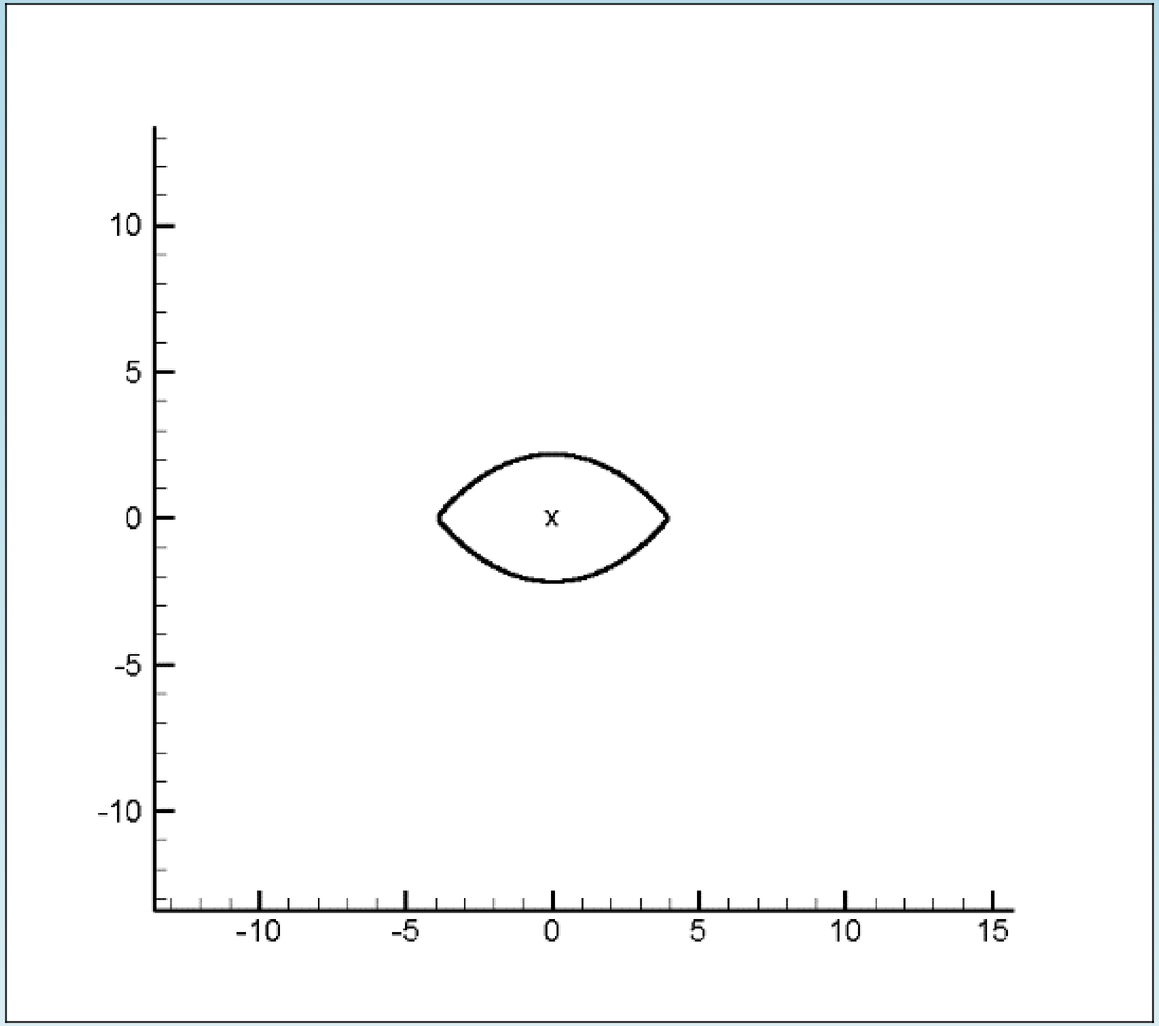
$$x_1 = 0,\ 2\left(S_0-\Omega_0\right)-\frac{\mu}{|y_1|}\left[\frac{1}{|y_1|}-K_1\left(|y_1|\right)\right]=0,$$

$$2\left(S_0+\Omega_0\right)+\frac{\mu}{|x_1|}\left[\frac{1}{|x_1|}-K_1\left(|x_1|\right)\right]=0,\ y_1=0.$$

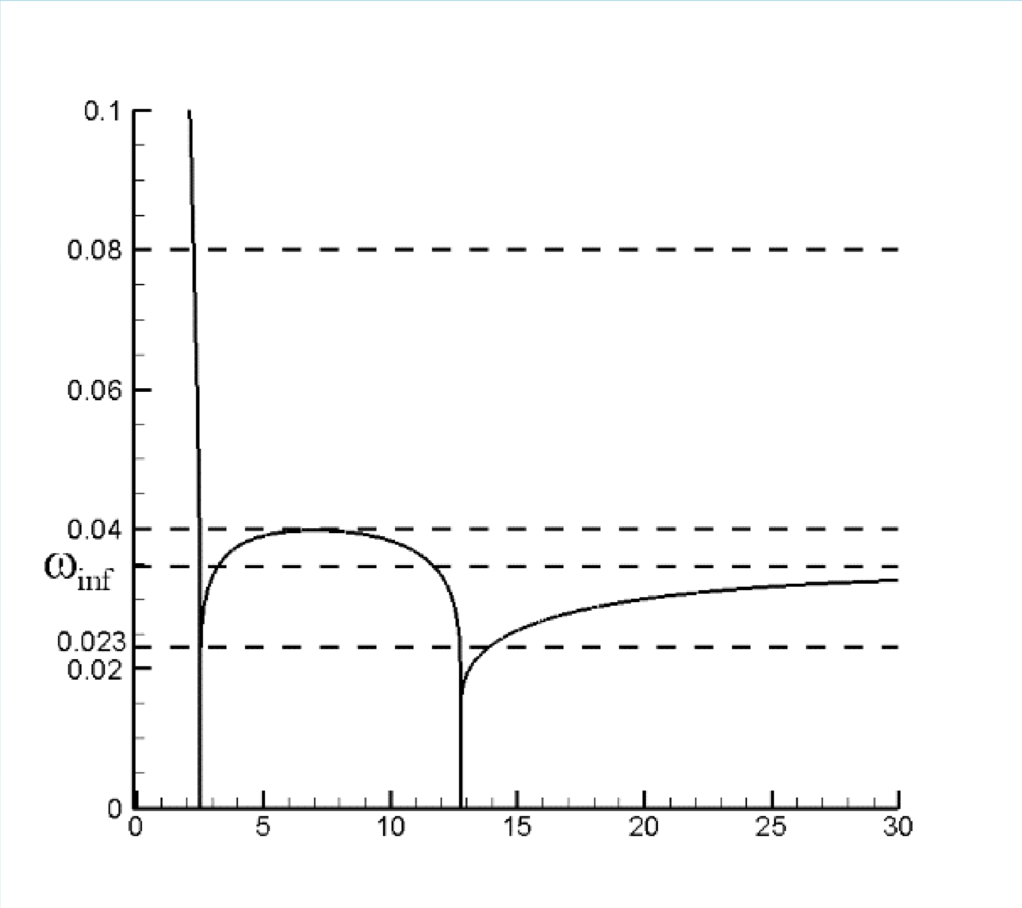


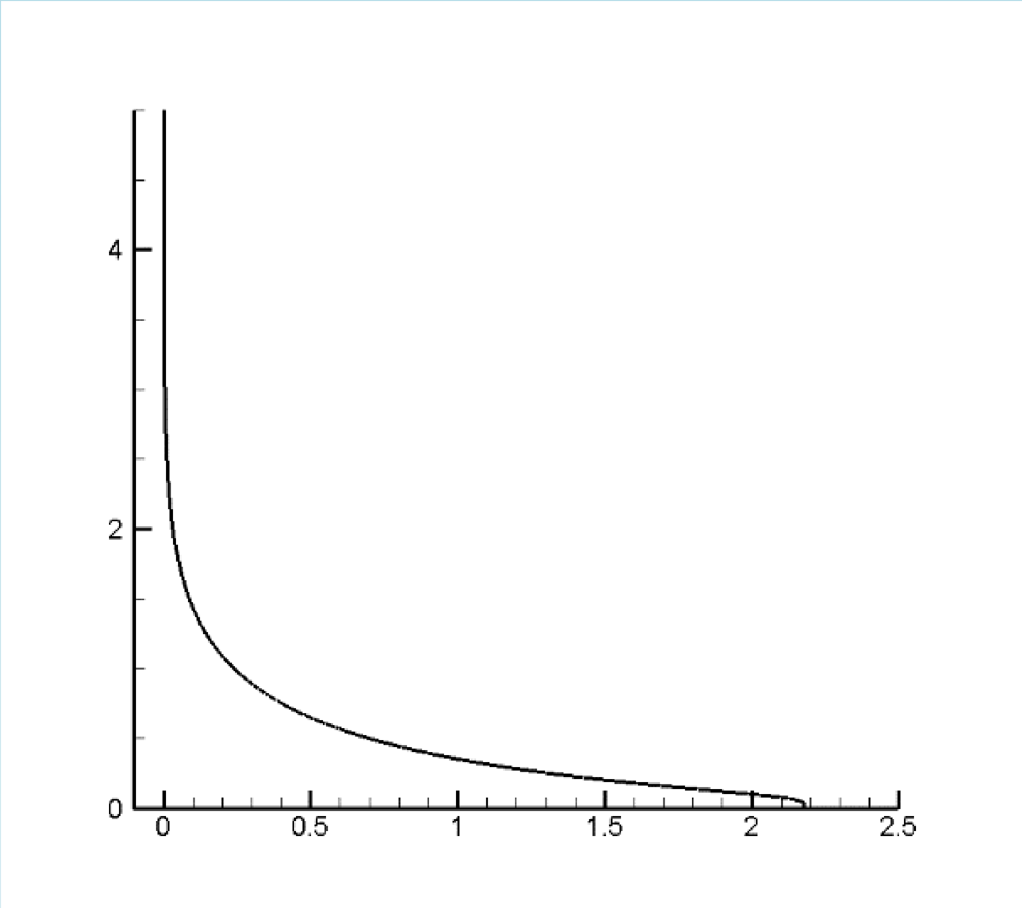
The diagram of the number of critical points and their types. 1 - two critical points on the y-axis, 2 - on the x-axis. Minus - hyperbolic points, plus - elliptic. Grey area - steady-state bounded motion of the vorticity center, white - unbounded.





$$S(t) = S_0(1 + \varepsilon_1 \sin \nu t),\ \Omega(t) = \Omega_0(1 + \varepsilon_2 \sin \nu t).$$

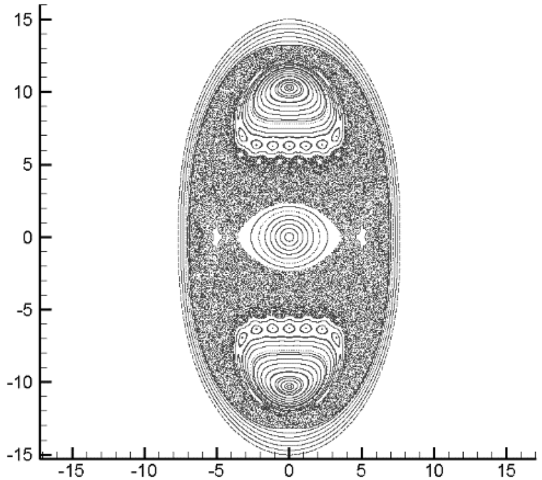




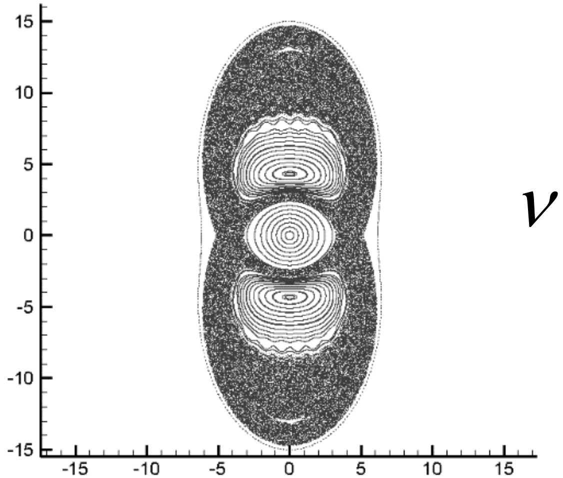
Eigen-frequencies of the vortex rotation about the elliptic points of phase portlets above, respectively. The dashed lines correspond to the perturbation frequencies used further. $\omega_{\rm inf} = 2\sqrt{\left|S_0{}^2 - \Omega_0{}^2\right|}$ is the frequency with which the vortices rotate about the vorticity center at infinity.

Poincare sections as $\mu = 1,\ S_0 = -0.01,\ \Omega_0 = -0.02$ $\quad \left(\left| (S_0+\Omega_0)/(S_0-\Omega_0) \right| |\varepsilon_1| = 3|\varepsilon_1| \right)$

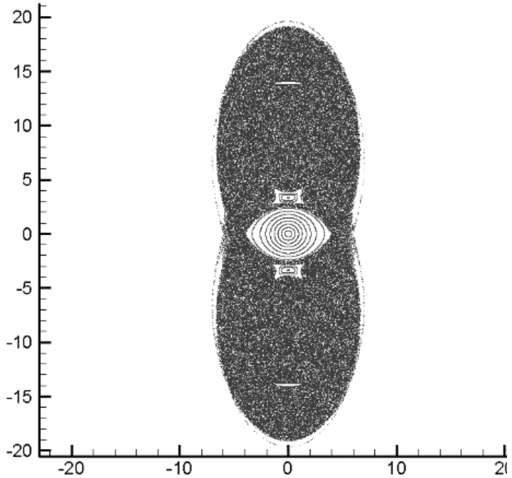
$\nu = 0.04$



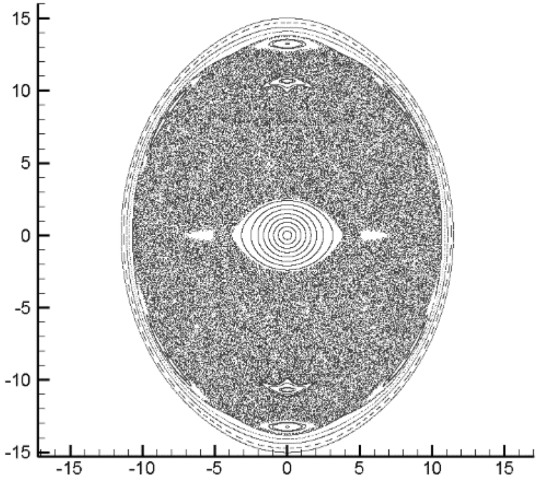
$\varepsilon_1 = \varepsilon_2 = 0.1$



$\varepsilon_1 = \varepsilon_2 = -0.1$

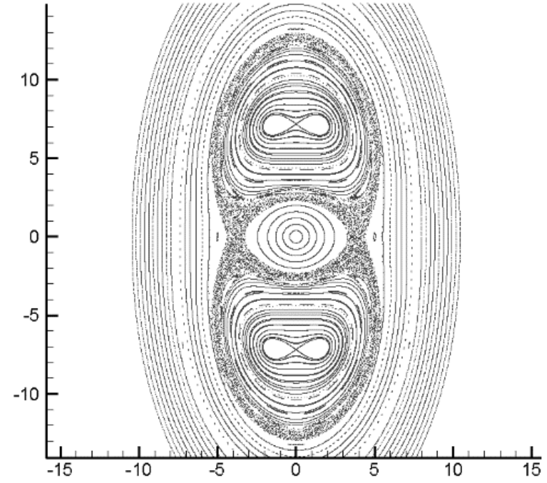


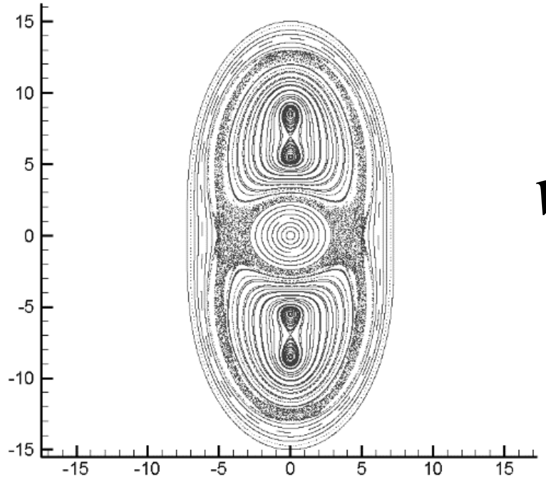
$\varepsilon_1 = 0.1,\ \varepsilon_2 = -0.1$

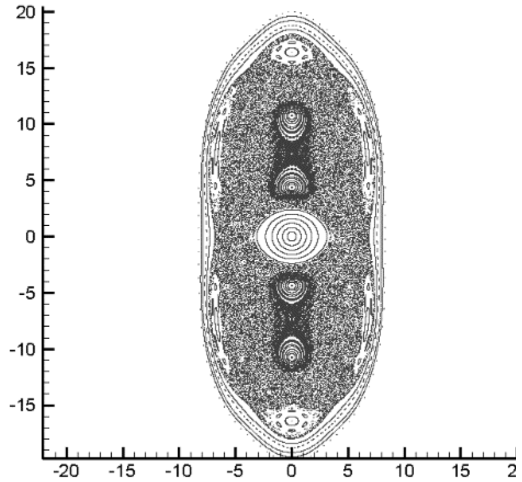


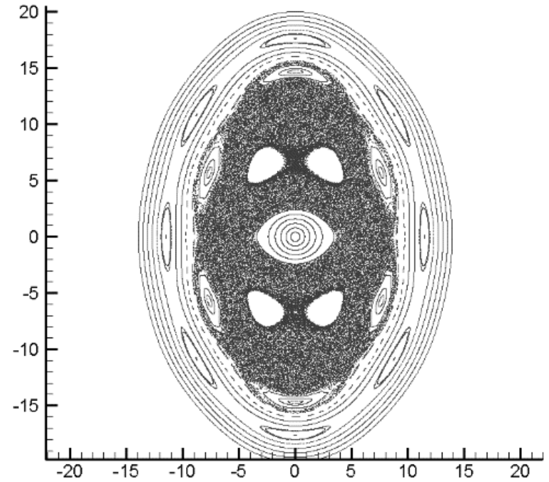
$\varepsilon_1 = -0.1,\ \varepsilon_2 = 0.1$

$\nu = 0.08$

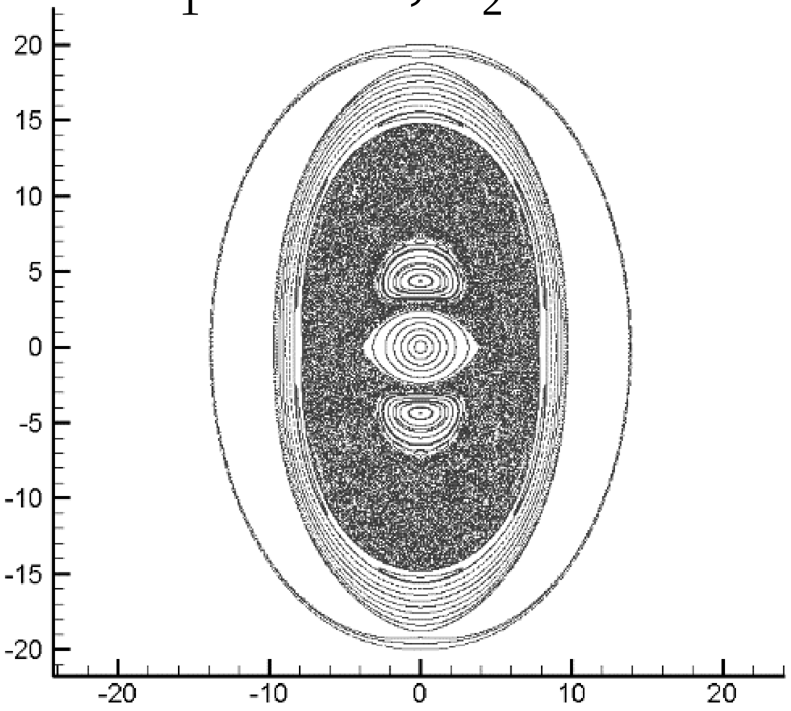




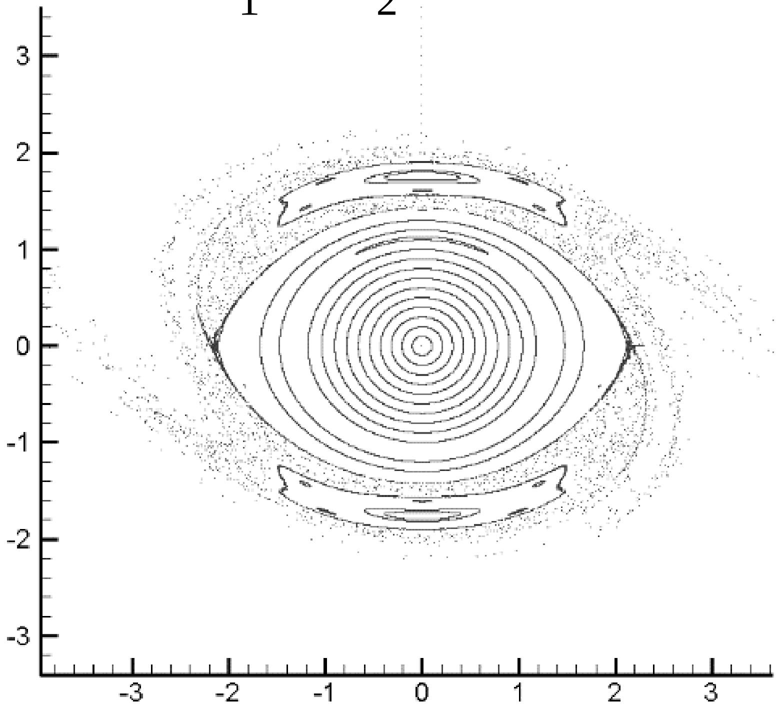


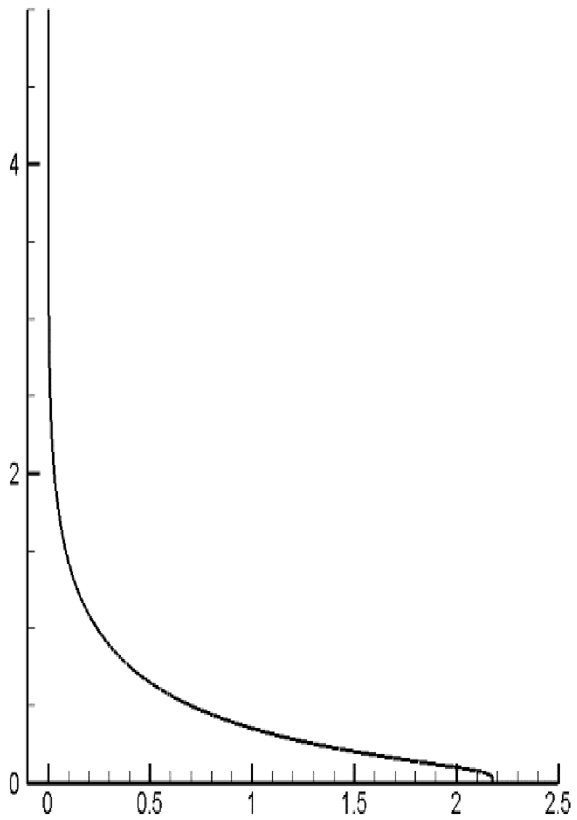


$\varepsilon_1 = -0.1,\ \varepsilon_2 = 0.1$



$\varepsilon_1 = \varepsilon_2 = 0.3$





$S_0 = -0.01, \Omega_0 = -0.02,$

$\mu = 1, \nu = 0.02$

$S_0 = -0.02, \Omega_0 = -0.01,$

$\mu = 1, \nu = 0.03$

III. Parametric instability of a two point-vortex system relative motion in a multi-layer flow under linear deformation near the elliptic point.

$$\frac{dx_1}{dt} = y_1\left(2\left(S(t) - \Omega(t)\right) - \frac{\mu}{r_1}\left[\frac{1}{r_1} - K_1(r_1)\right]\right),$$

$$\frac{dy_1}{dt} = x_1\left(2\left(S(t) + \Omega(t)\right) + \frac{\mu}{r_1}\left[\frac{1}{r_1} - K_1(r_1)\right]\right).$$

$$\frac{dx_1}{dt} = y_1\left(2\left(S(t) - \Omega(t)\right) - \frac{\mu}{r_1}\left[\frac{1}{r_1}\right]\right),$$

$$\frac{dy_1}{dt} = x_1\left(2\left(S(t) + \Omega(t)\right) + \frac{\mu}{r_1}\left[\frac{1}{r_1}\right]\right).$$

$$\frac{dx}{dt} = 2y_1^0 \left(S_0 - \Omega_0\right)\varepsilon \sin \nu t + 2\left(\left(S_0 - \Omega_0\right)\varepsilon \sin \nu t + \frac{\mu_2}{\left(y_1^0\right)^2} K \right) y,$$

$$\frac{dy}{dt} = 2x\left(2S_0 + \left(S_0 + \Omega_0\right)\varepsilon \sin \nu t\right),$$

$$\frac{dX}{dt} = A(t) + 2Y\left(S(t) - \Omega(t)\right),$$

$$\frac{dY}{dt} = -B(t) + 2X\left(S(t) + \Omega(t)\right).$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + 2y_1^0\left(S_0 - \Omega_0\right)\varepsilon \mathbf{X}(t)\int_{t_0}^{t} \mathbf{X}^{-1}(\tau)\begin{pmatrix} \sin \nu\tau \\ 0 \end{pmatrix} d\tau,$$

$$\frac{dx}{dt} = 2y\left(2\left|S_0 - \Omega_0\right|K + \left(S_0 - \Omega_0\right)\varepsilon\sin\nu t\right),$$

$$\frac{dy}{dt} = 2x\left(2S_0 + \left(S_0 + \Omega_0\right)\varepsilon\sin\nu t\right).$$

$$R = i\frac{x}{y} - 1,$$

$$\frac{dR}{dt} = i\left(a_1 + a_2\sin\nu t\right) + i\left(b_1 + b_2\sin\nu t\right)\left(R+1\right)^2,$$

$$R = \rho(t)e^{-i\nu t},$$

$$\frac{d\rho}{dt} - i\nu\rho = i\left(a_1 + b_1\right)e^{i\nu t} + \frac{a_2 + b_2}{2}\left(e^{2i\nu t} - 1\right) +$$

$$+ib_1\left(2\rho + \rho^2 e^{-i\nu t}\right) - \frac{b_2}{2}e^{-i\nu t}\left(2\rho + \rho^2 e^{i\nu t}\right) + \frac{b_2}{2}\left(2\rho e^{i\nu t} + \rho^2\right).$$

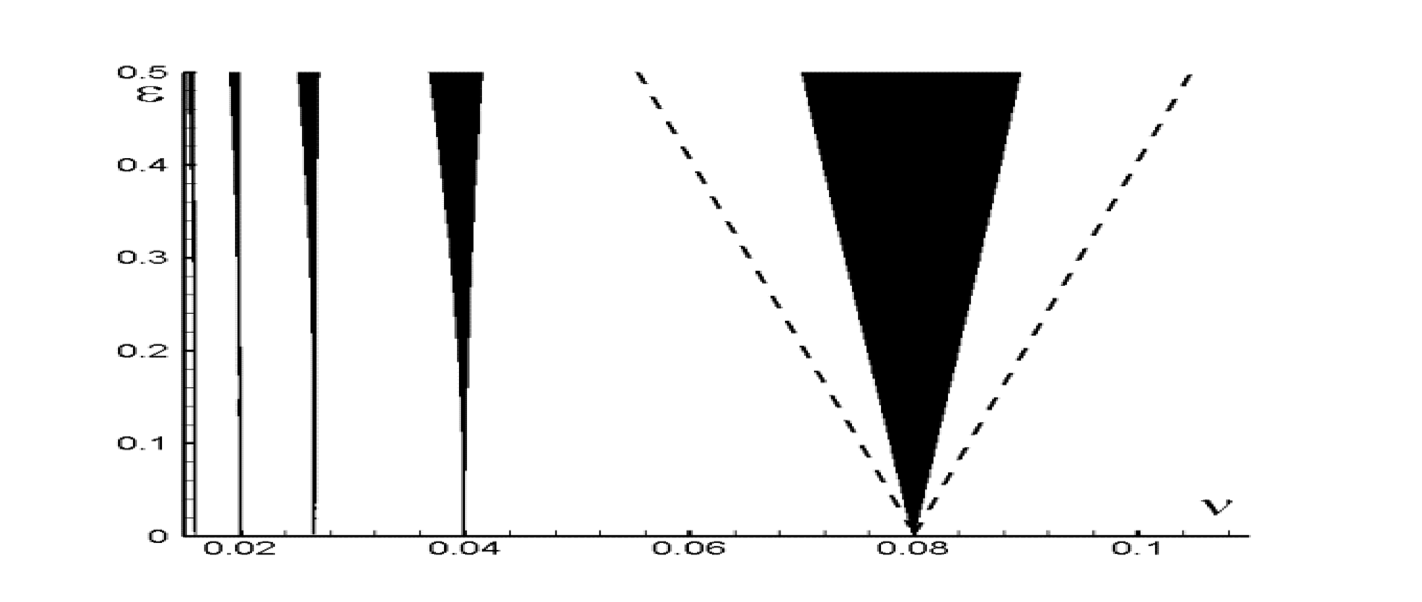
$$\frac{d\bar{\rho}}{dt} - i\nu\bar{\rho} = -\frac{a_2+b_2}{2} + 2ib_1\bar{\rho} + \frac{b_2}{2}\bar{\rho}^2, \quad \bar{\rho}(0) = 0.$$

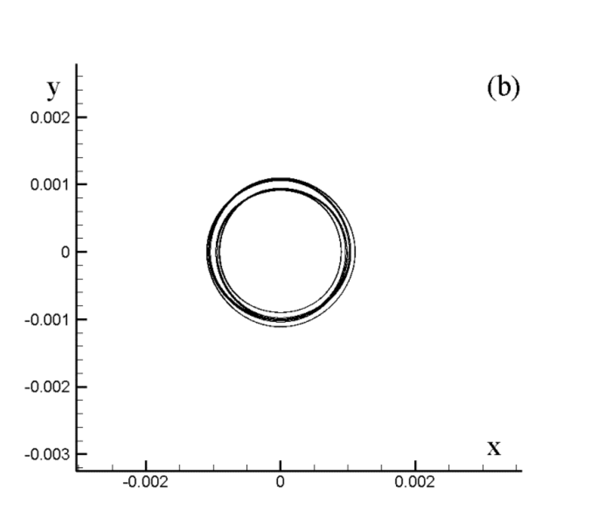
$$\frac{\bar{\rho}-\rho_1}{\bar{\rho}-\rho_2} = e^{\frac{b_2}{2}(\rho_2-\rho_1)t}, \qquad \rho_{1,2} = \pm\left(\frac{a_2+b_2}{b_2} - \left(\frac{2b_1+\nu}{b_2}\right)^2\right)^{1/2} + \frac{i(2b_1+\nu)}{2},$$

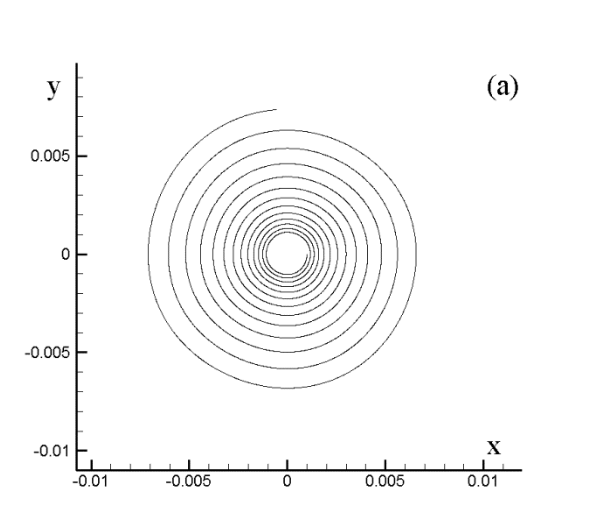
$$\rho_2 - \rho_1 = \pm 2\left(\frac{a_2+b_2}{b_2} - \left(\frac{2b_1+\nu}{b_2}\right)^2\right)^{1/2}, \quad \rho_2 - \rho_1 = \pm 2\left(\frac{2S_0}{(S_0+\Omega_0)} - \left(\frac{8S_0+\nu}{2\varepsilon(S_0+\Omega_0)}\right)^2\right)^{1/2}.$$

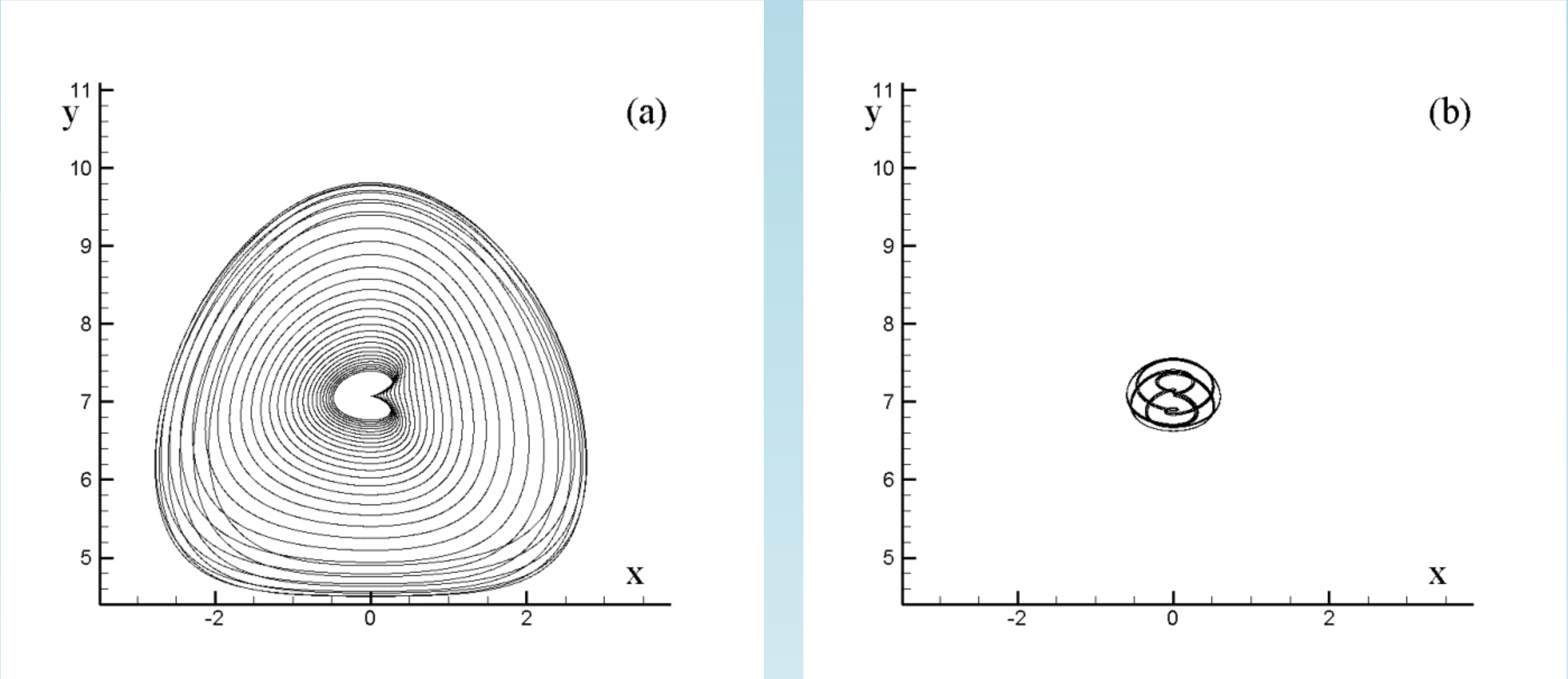
$$\det(\mathbf{X}(T) - \mathbf{E}\rho) = \det\begin{pmatrix} x_1(T)-\rho & x_2(T) \\ y_1(T) & y_2(T)-\rho \end{pmatrix} = \rho^2 - (x_1(T)+y_2(T))\rho + 1 = 0,$$

$$x_1(T;\nu,\varepsilon) + y_2(T;\nu,\varepsilon) = \pm 2, \qquad T = 2\pi/\nu$$

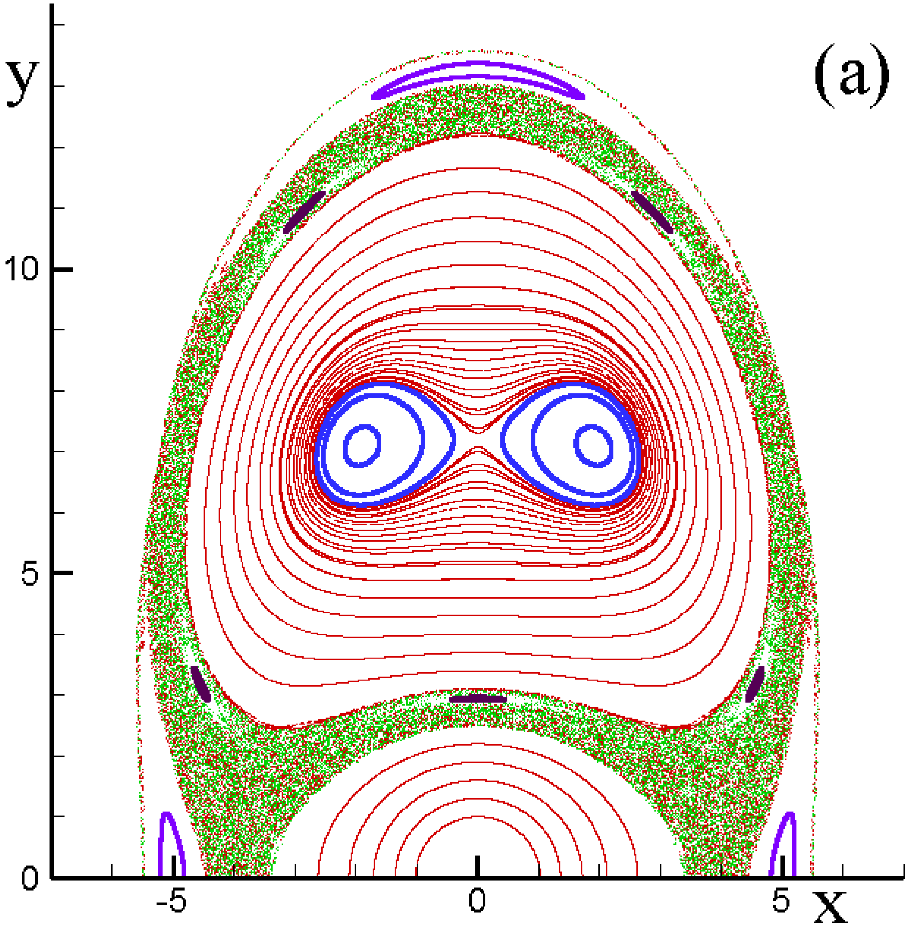

0.5
ε
0.4
0.3
0.2
0.1
0
0.02
0.04
0.06
0.08
0.1
ν

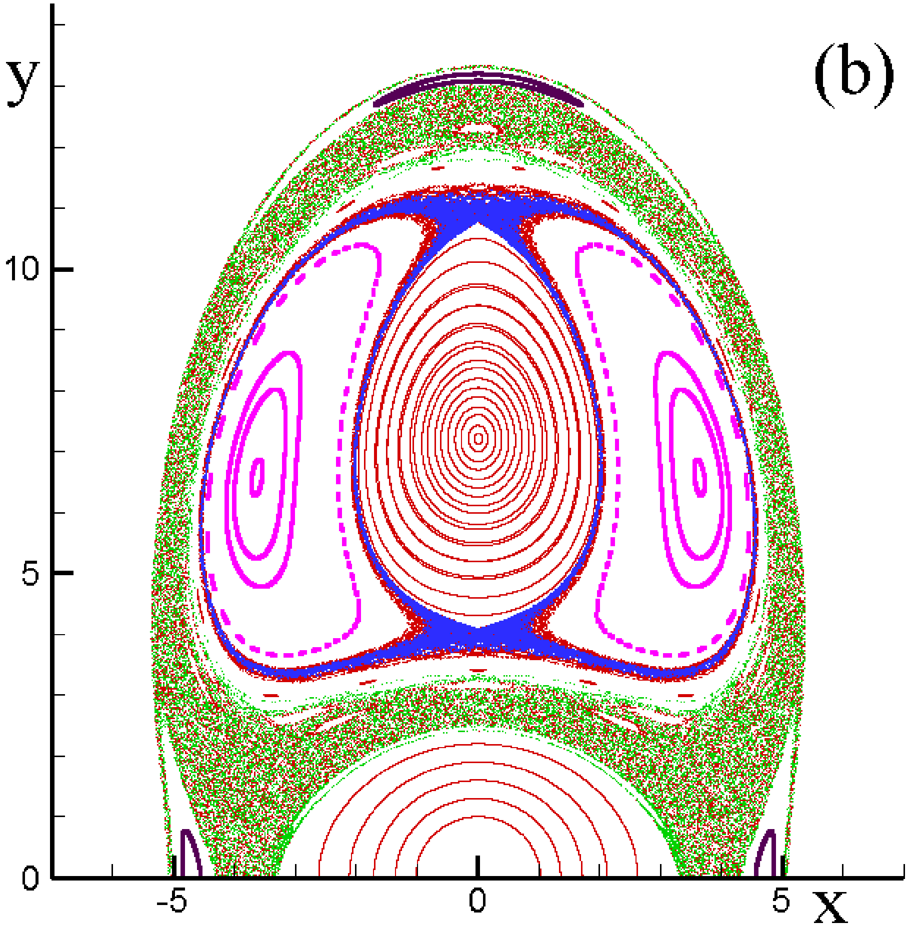

y
(b)
0.002
0.001
0
-0.001
-0.002
-0.003
x
-0.002
0
0.002

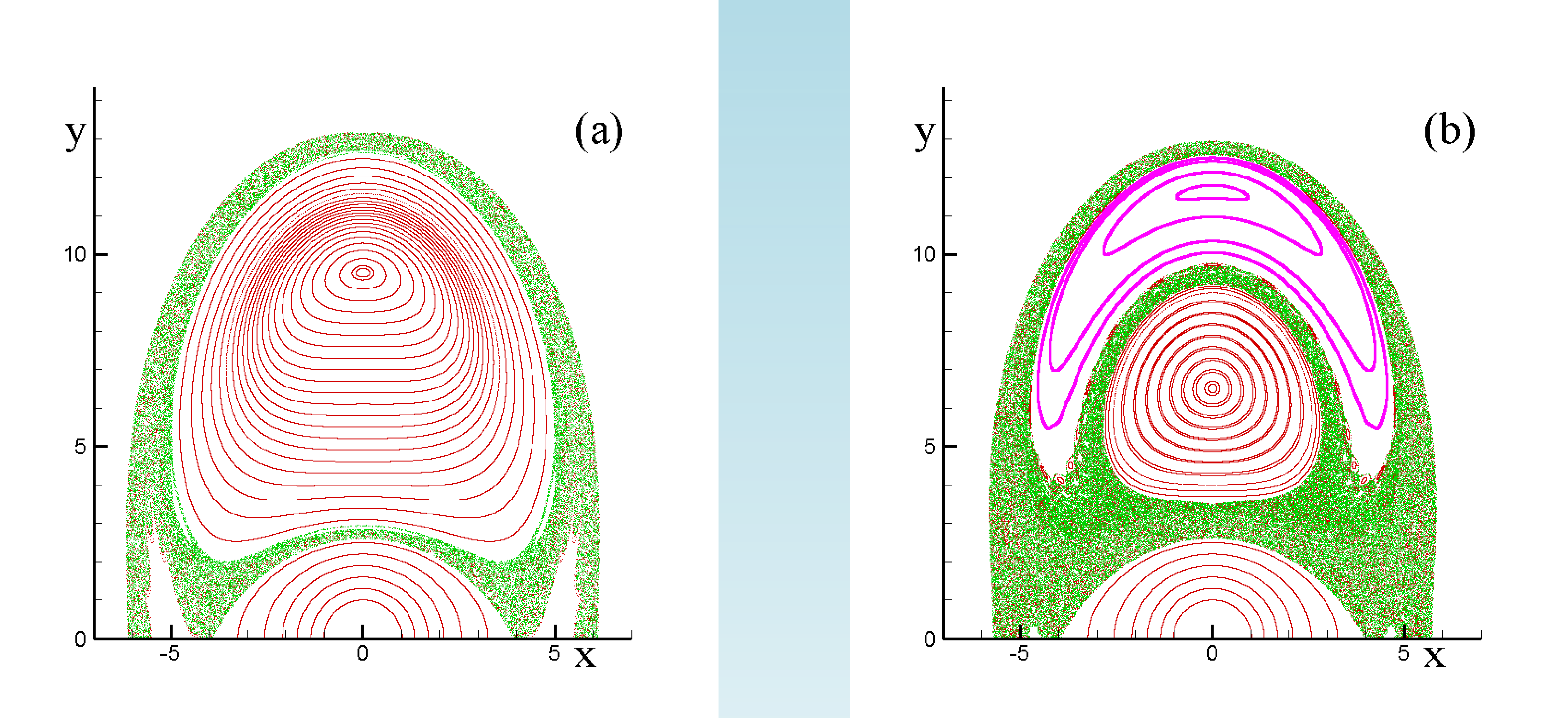

y
(a)
0.005
0
-0.005
-0.01
x
-0.01
-0.005
0
0.005
0.01



Trajectories of the original system for S 0 =0.01, Ω_0 = -0.02. (a) locally unbounded motion attributed to the parametric instability for ν= 0.08; ε= 0:1; (b) locally bounded motion with no parametric instability for ν= 0.07; ε= 0:1.


y
(a)
10
5
0
-5
0
5
x


y
(b)
10
5
0
-5
0
5
x


(a)
y
10
5
0
-5
0
5
x
(b)
y
10
5
0
-5
0
5
x

Thanks for You attention!

Thanks for **Organizing Committee!**

And special Thanks for professor

Ana M. Mancho!