Image Based Analysis of Coherency Directly From Movies

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and with Basaynake

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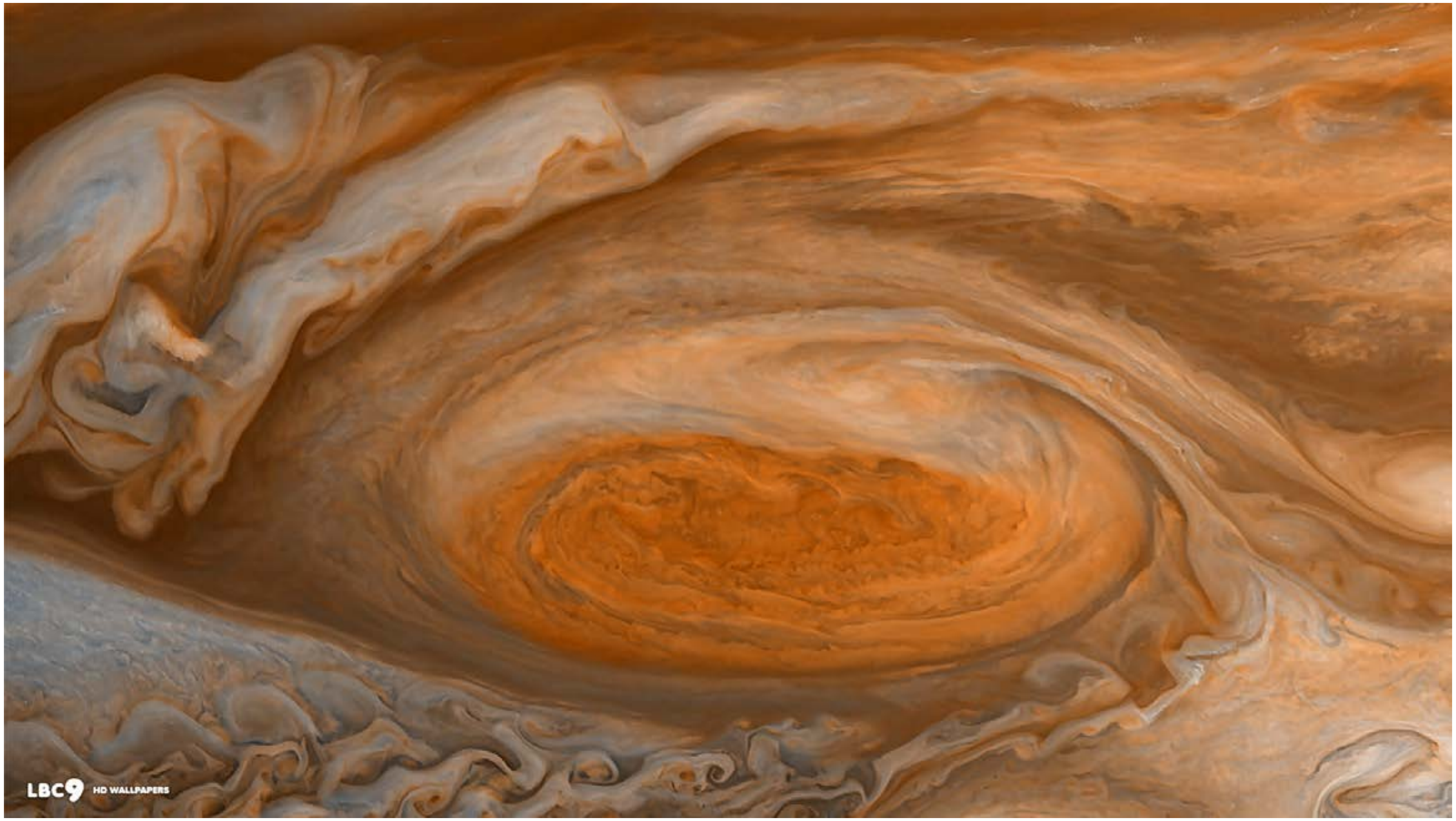
Jupiter Portrait as viewed from Cassini. "This true color mosaic of Jupiter was constructed from images taken by the narrow angle camera onboard NASA's Cassini spacecraft on December 29, 2000, during its closest approach to the giant planet at a distance of approximately 10 million kilometers (6.2 million miles)."[1].

Clearly, time scale and space scale are each important regarding questions concerning some sets "hold together"

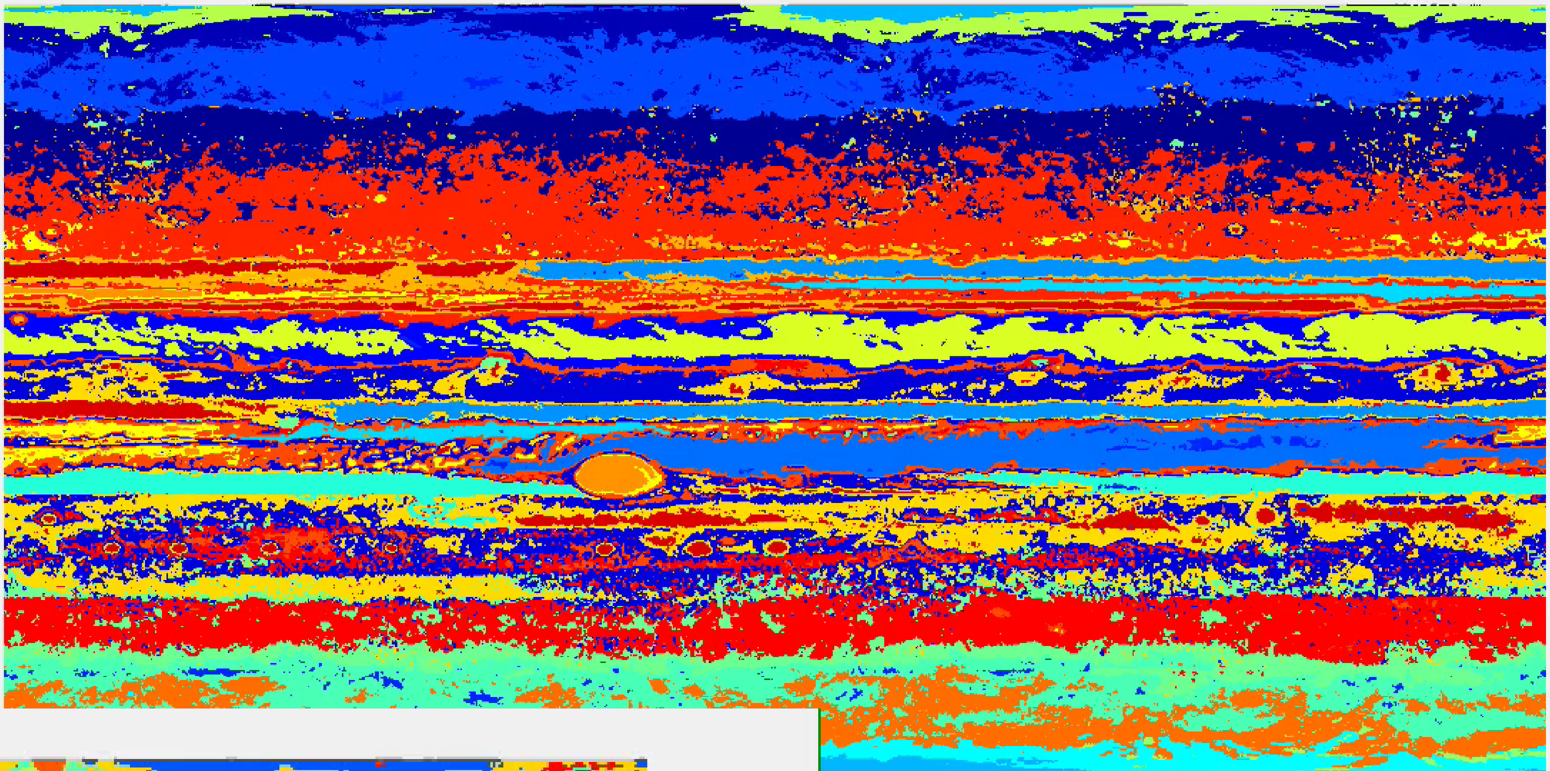


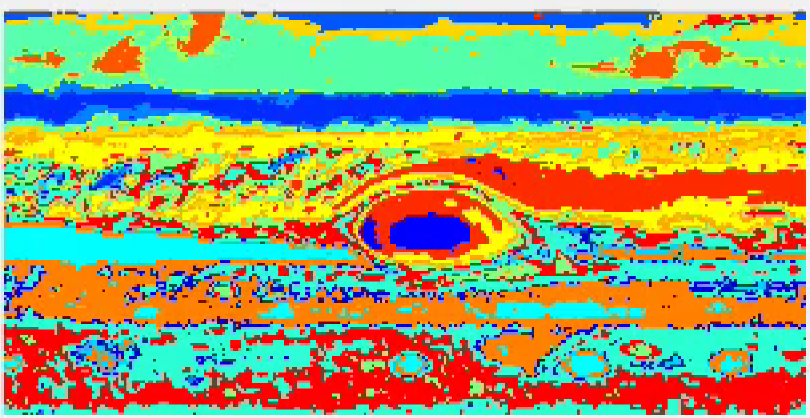
Iconic Coherent Set

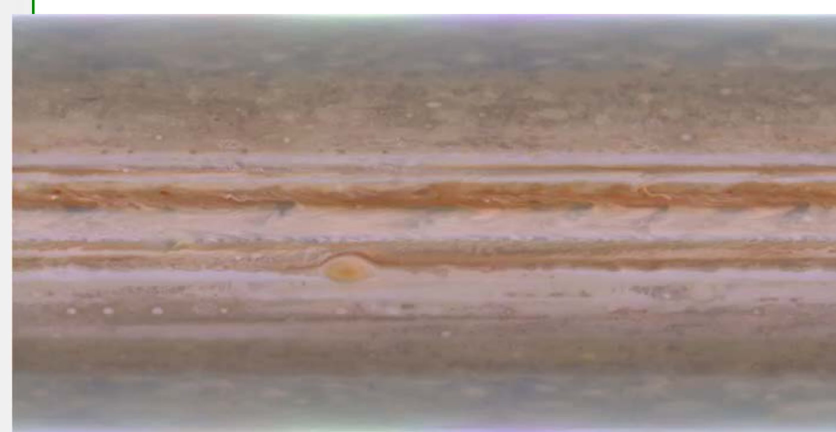
On certain time scale and space scale, some sets *"hold together."*

Many definitions of coherency, and corresponding methods,

A punch-line first – Looking for a “Motion Segmentation” without a vector field







Pondering What to Call Coherence
– Coherence Means A lot of things to A lot of People

-Follow the density? (Ensembles of initial conditions).
….flux or transport of density in and out of sets? (Transfer Operator Methods)

-Follow measurements - measurables? (And the Koopman methods)
(or combination).
-Follow strain of Boundaries? (LCS methods)

-Match shape? Exactly? Approximately? (=> Shape Coherence=> FTC)

Should there be a unique solution to the concept of coherence?
LCS? Coherent Pair? Mesohyperbolic? Shape Coherent? Exact Coherent?

-A clustering method

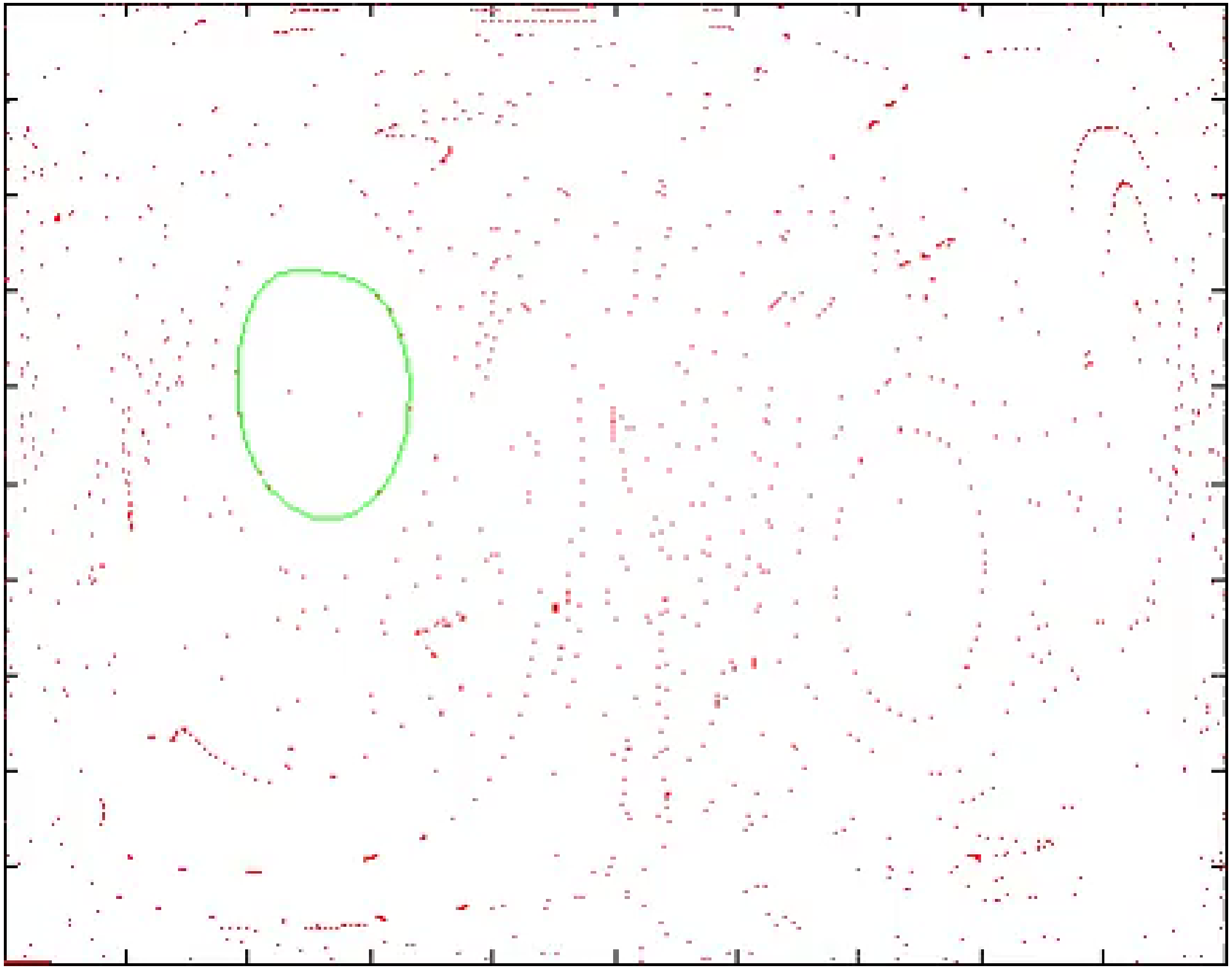
-Follow the "surprise" – information theory story

ALL of these need a step to develop a model as SOME WAY to follow particles – a flow map, a vector field, etc.

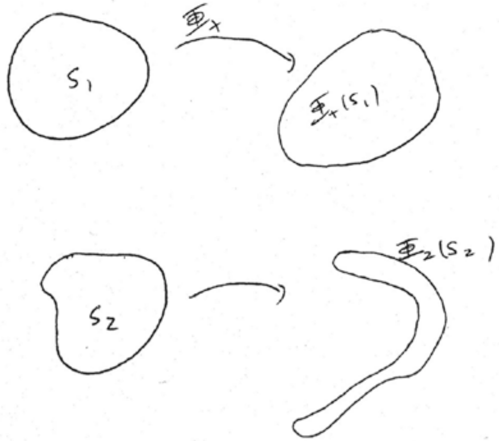
What can we do DIRECTLY from watching a movie?

From my previous discussion **- Sets that keep their shape "look" coherent.**

The Green Set is Such a Set



Tian Ma and Erik M. Bollt, "Differential Geometry Perspective of Shape Coherence and Curvature Evolution by Finite-Time Nonhyperbolic Splitting, (SIADS), 2014

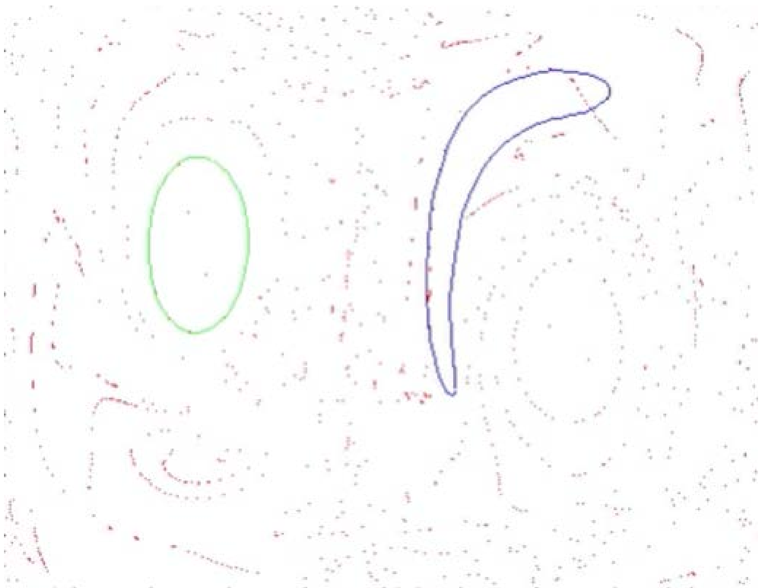


Tian Ma and Erik M. Bollt, "Differential Geometry Perspective of Shape Coherence and Curvature Evolution by Finite-Time Nonhyperbolic Splitting, (SIADS), 2014

Coherence vs continuity

– image of a connected set is connected

Before



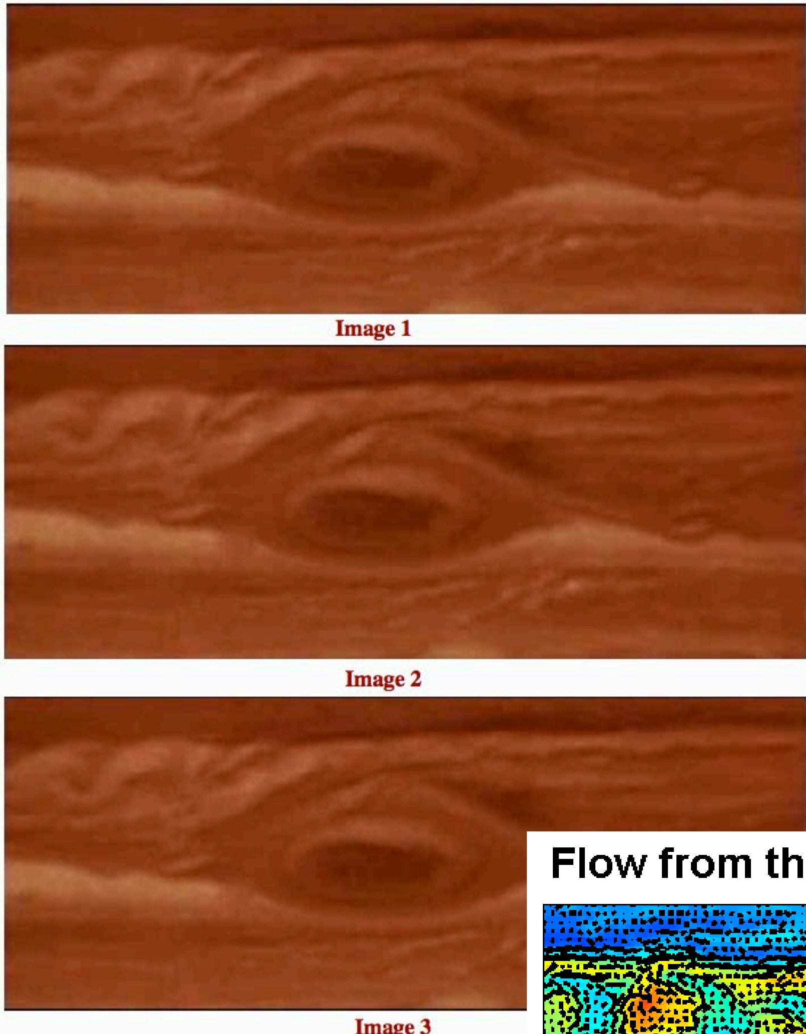
After

The Green Set is Such a Set

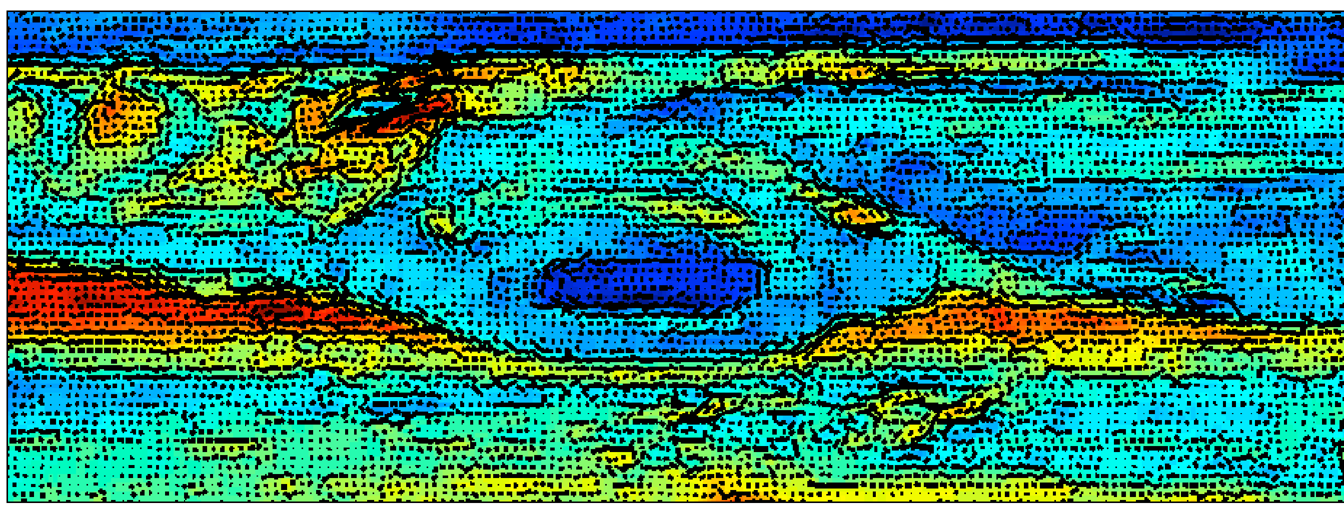
-Froyland, Junge
-G. Haller & F.J. Beron-Vera

Modelling the vector field of the flow from images

-again - can we skip this step?

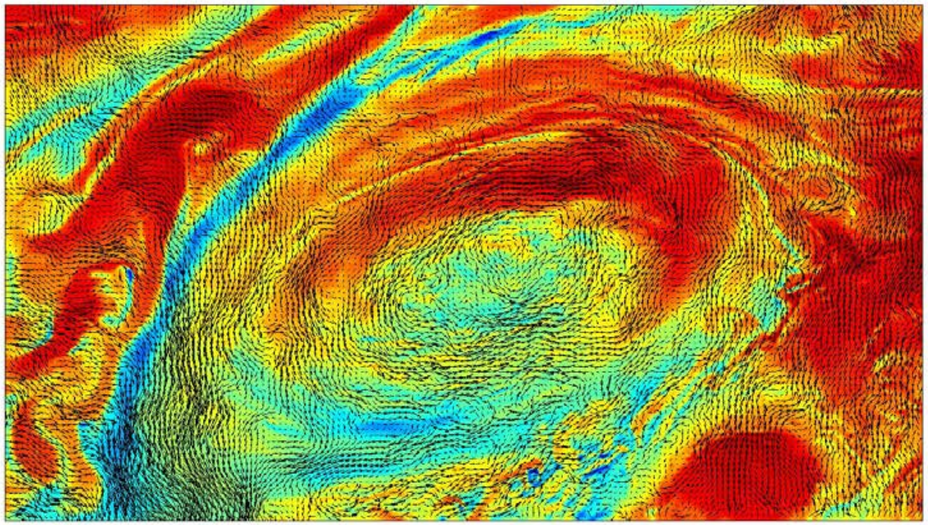


Flow from the Quasi-Geostrophic Multi-time step method with n=2



Basaynake, Bollt

In an Effort as
SOME WAY to follow particles



$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t),$$

$$\mathbf{F} : M \to M$$

a flow mapping, $\mathbf{x}(t) = \Phi(\mathbf{x}_0, t_0, t)$ is inferred

OPTICAL FLOW METHOD

Classical (UV) method

- The image brightness, $I(x, y, t)$ of a point (x, y) is locally conservative over the time t.
- The problem is ill-posed and hence regularized by assuming the expected flow is smooth.
- The energy functional is

$$E(u,v) = \int_\Omega \left(I_t + I_x u + I_y v\right)^2 d\Omega + \alpha \int_\Omega \left(u_x^2 + u_y^2 + v_x^2 + v_y^2\right) d\Omega$$

Stream function method

- If the stream function is ψ then $\langle u, v\rangle = \langle -\psi_x, \psi_y\rangle$
- The new Energy functional,

$$E(\psi) = \int_\Omega (I_t - I_x\psi_y + I_y\psi_x)^2 d\Omega + \alpha \int_\Omega (\psi_{xx}^2 + \psi_{xy}^2 + \psi_{yx}^2 + \psi_{yy}^2)\, d\Omega$$

- The Euler-Lagrange equation is

$$[A^*A + \alpha(B + B^*)]\psi = -A^* I_t$$

where $A = -I_x D_y + I_y D_x$ and $B = D_{xx}^* D_{xx} + D_{xy}^* D_{xy} + D_{yx}^* D_{yx} + D_{yy}^* D_{yy}$

Multi-time step method

- Apply when a sequence of images is available
- Energy functional for two stream functions

$$E(\psi_1, \psi_2) = \int_\Omega (I_{1t} - I_{1x}\psi_{1y} + I_{1y}\psi_{1x})^2 + (I_{2t} - I_{2x}\psi_{2y} + I_{2y}\psi_{2x})^2 d\Omega$$

$$+\beta \int_\Omega (\psi_1 - \psi_2)^2 d\Omega +$$

$$\alpha \int_\Omega (\psi_{1xx}^2 + \psi_{1xy}^2 + \psi_{1yx}^2 + \psi_{1yy}^2 + \psi_{2xx}^2 + \psi_{2xy}^2 + \psi_{2yx}^2 + \psi_{2yy}^2)\, d\Omega$$

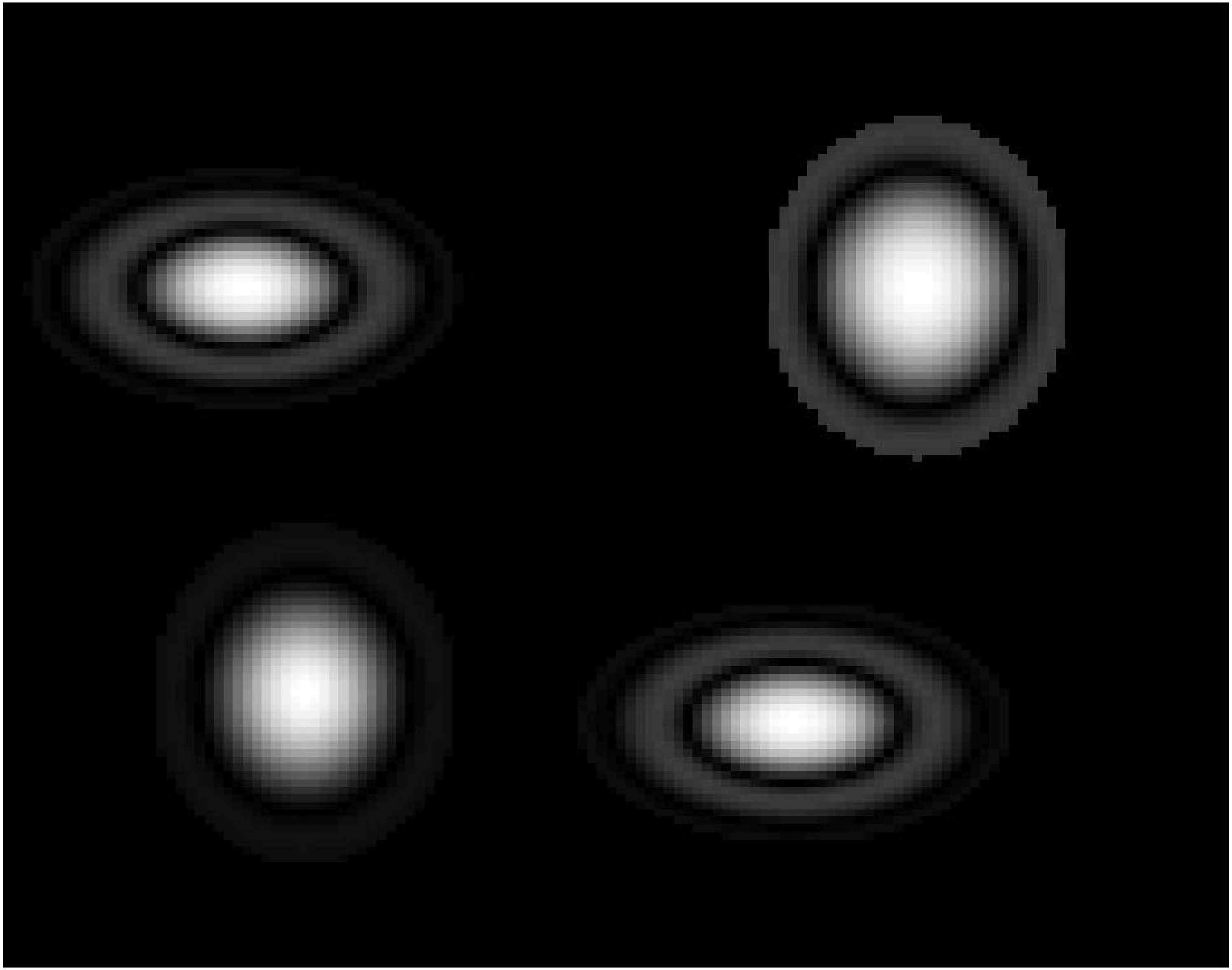
- Euler-Lagrange equations are

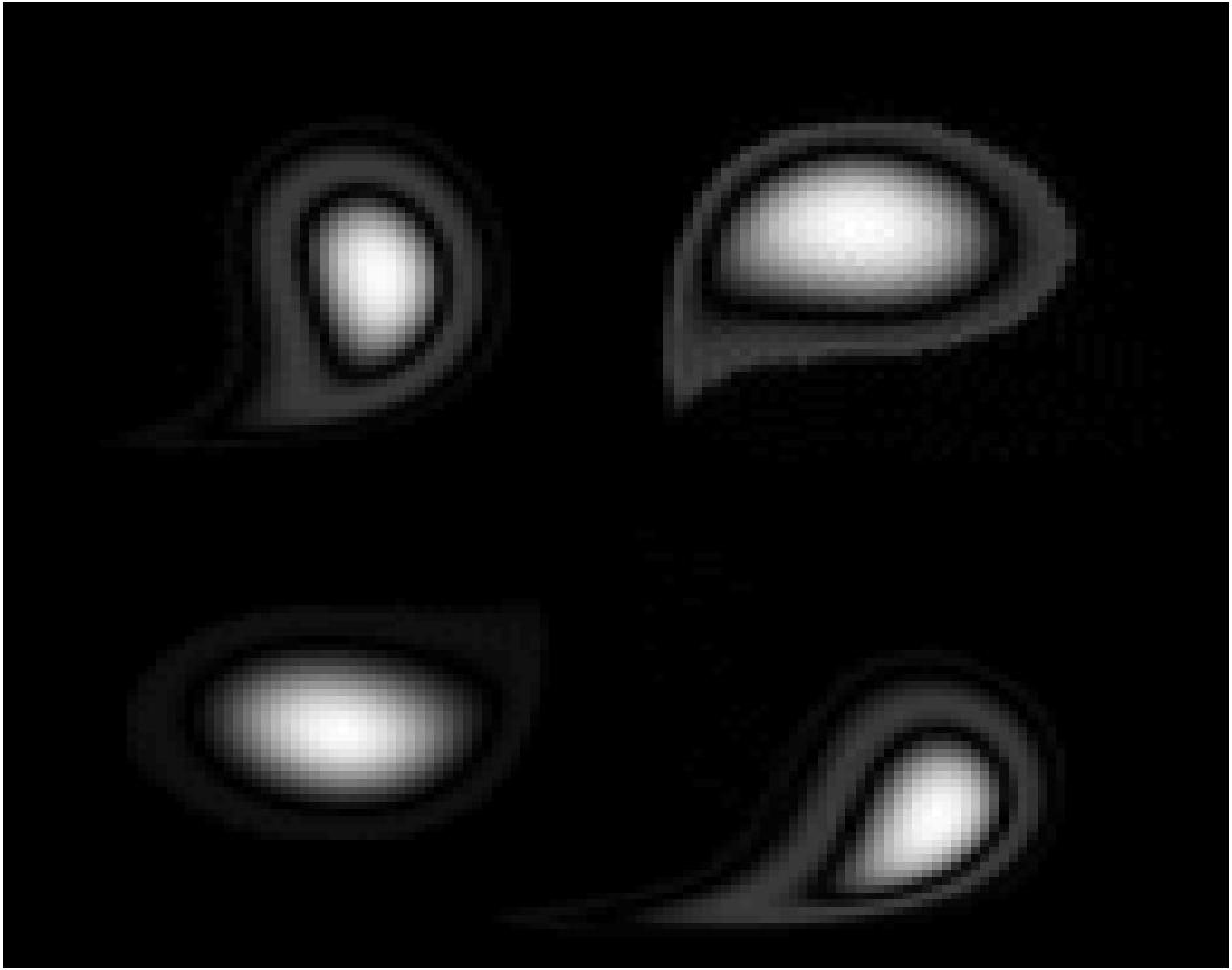
$$A_1^*(I_{1t} + A_1\psi_1) + \beta(\psi_1 - \psi_2) + \alpha B\psi_1 = 0$$
$$A_2^*(I_{2t} + A_2\psi_2) - \beta(\psi_1 - \psi_2) + \alpha B\psi_2 = 0$$

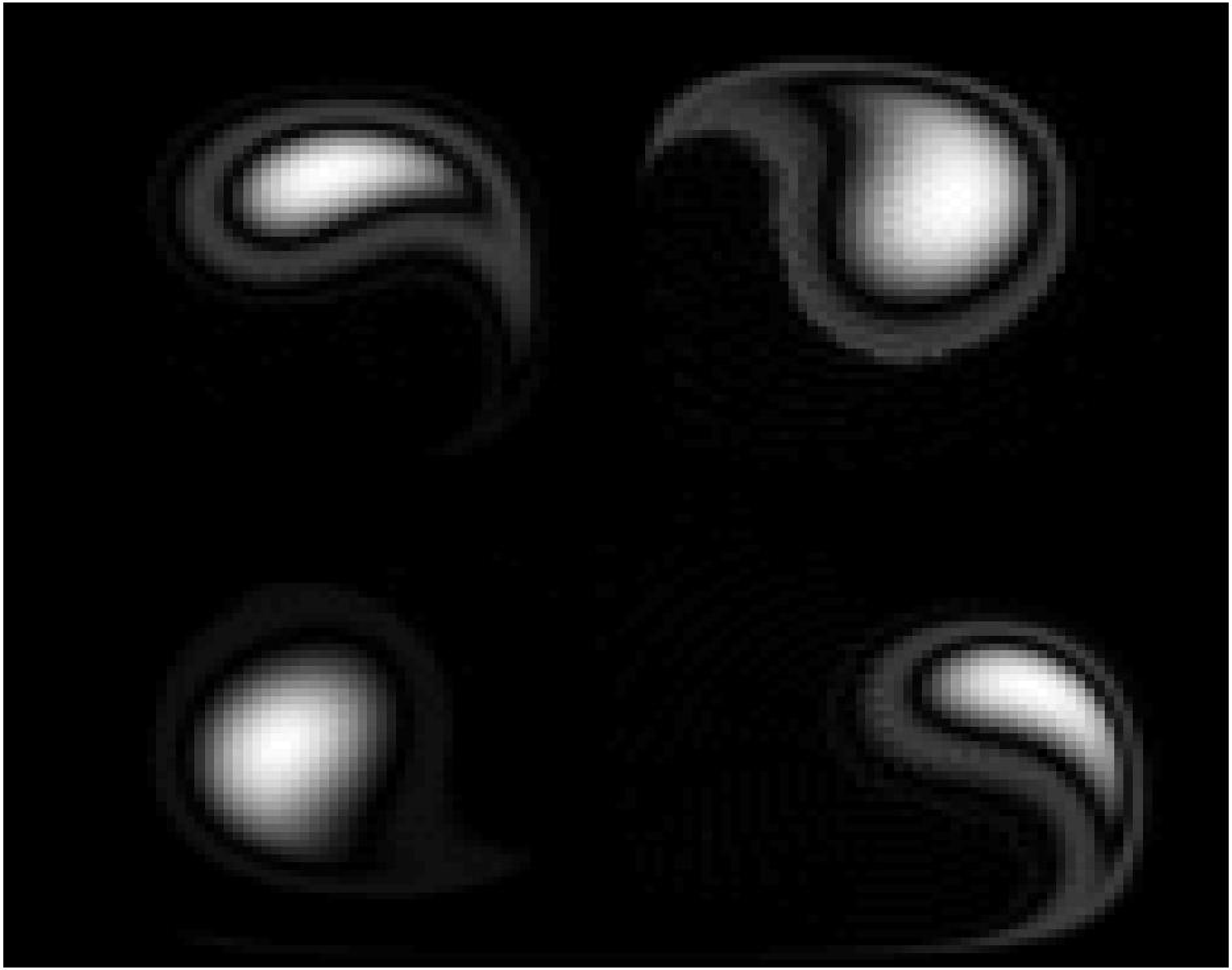
- Energy functional for n stream functions

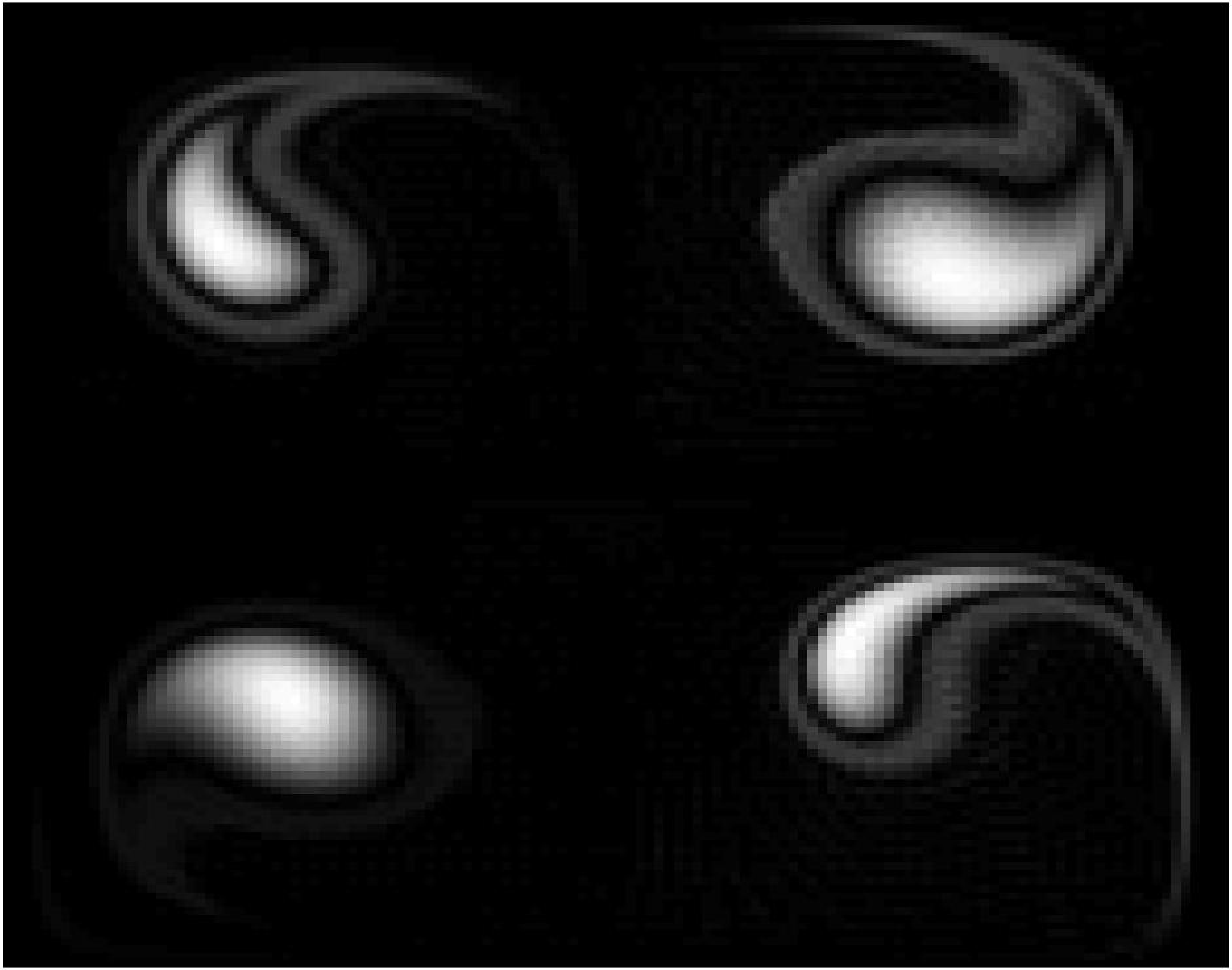
$$E(\psi_1, \psi_2, \ldots \psi_n)$$
$$= \sum_{k=1}^{n} \int_\Omega (I_{kt} - I_{kx}\psi_{ky} + I_{ky}\psi_{kx})^2 d\Omega + \beta \sum_{k=1}^{n-1} \int_\Omega (\psi_k - \psi_{k+1})^2 d\Omega$$
$$+ \alpha \sum_{k=1}^{n} \int_\Omega (\psi_{kxx}^2 + \psi_{kxy}^2 + \psi_{kyx}^2 + \psi_{kyy}^2)\, d\Omega$$

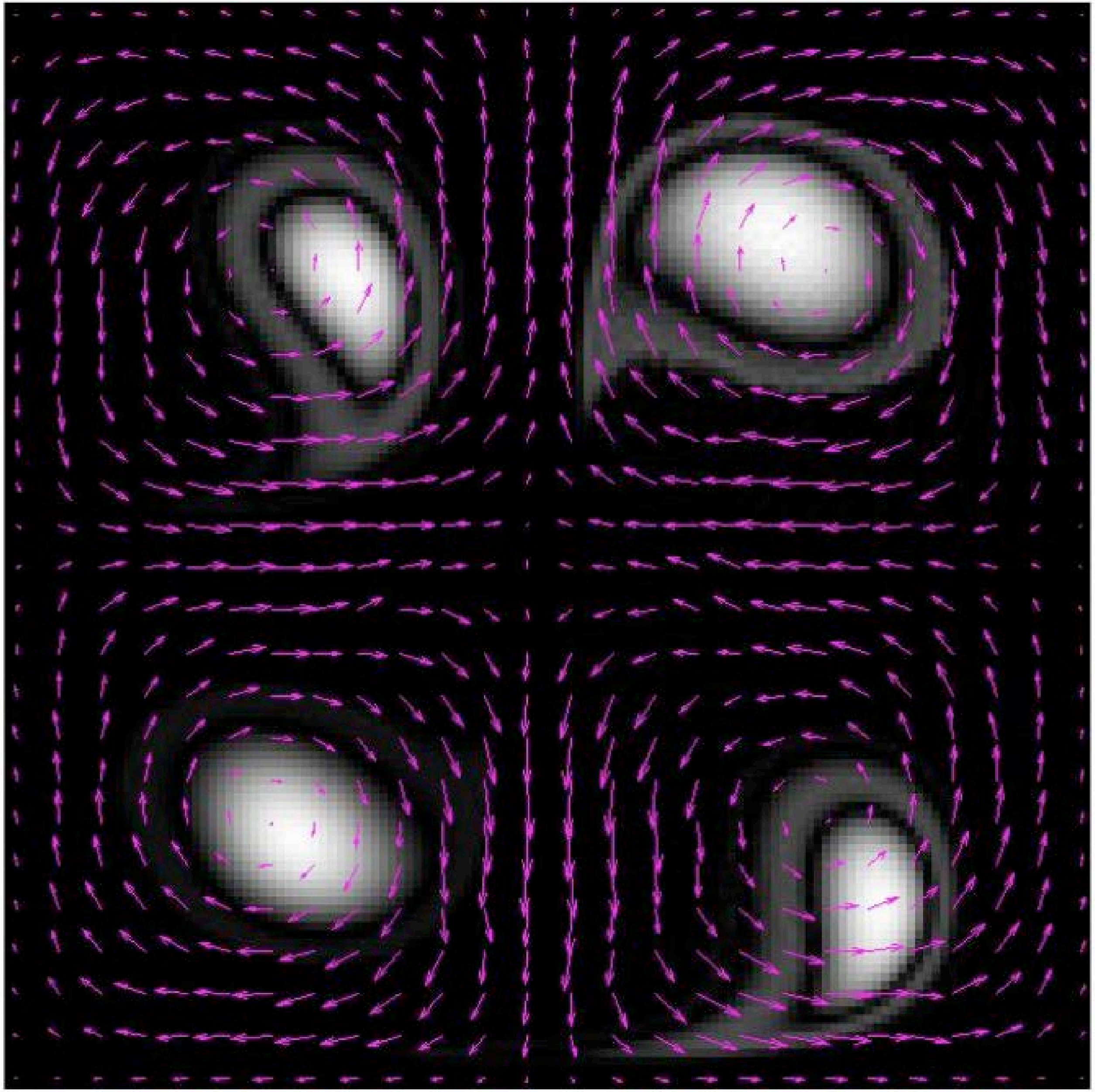
- Need to solve n linear Euler-Lagrange equations





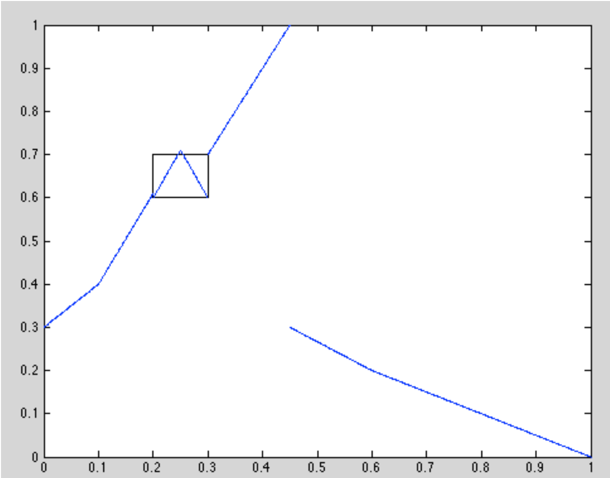




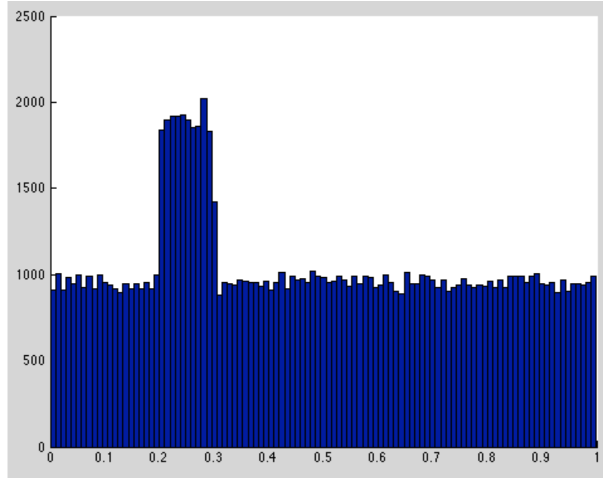


But do we really need the vector field?

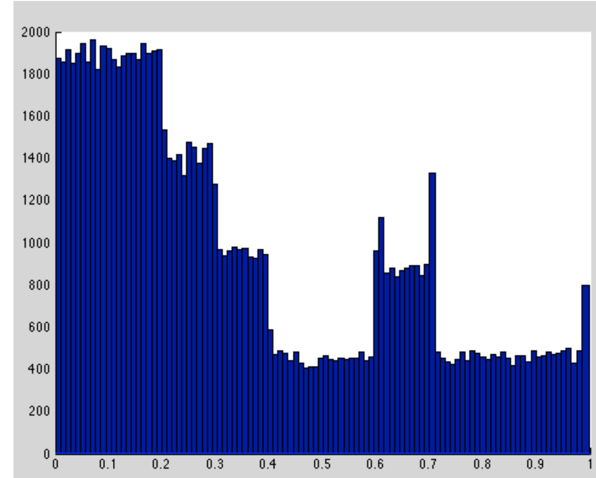
A hyperbolic coherent set vs an elliptic coherent set



T(x)

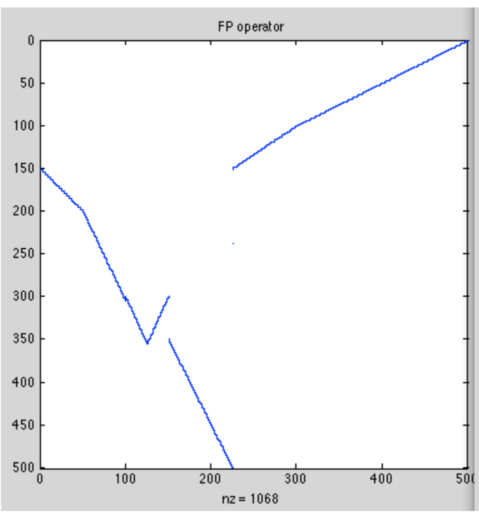


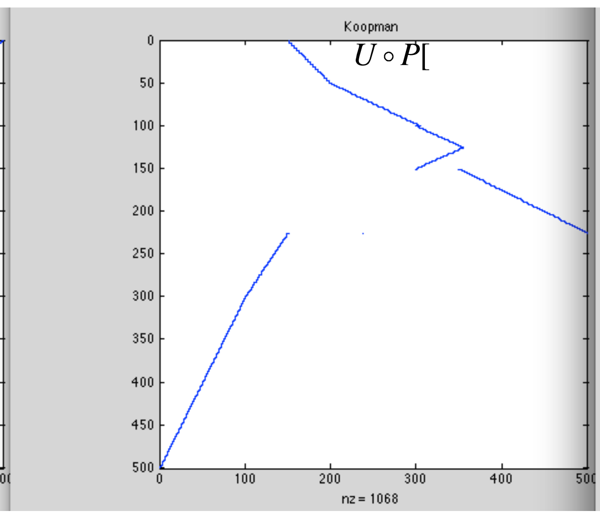
r0(x)

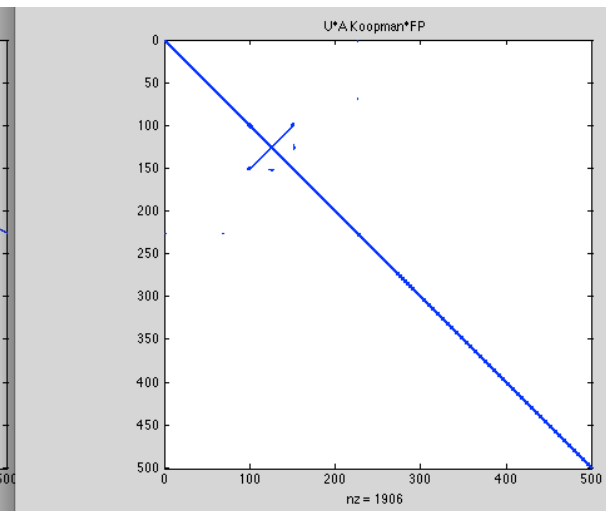


r1(x)

All the particles move together BUT they mix/shuffle within the coherent







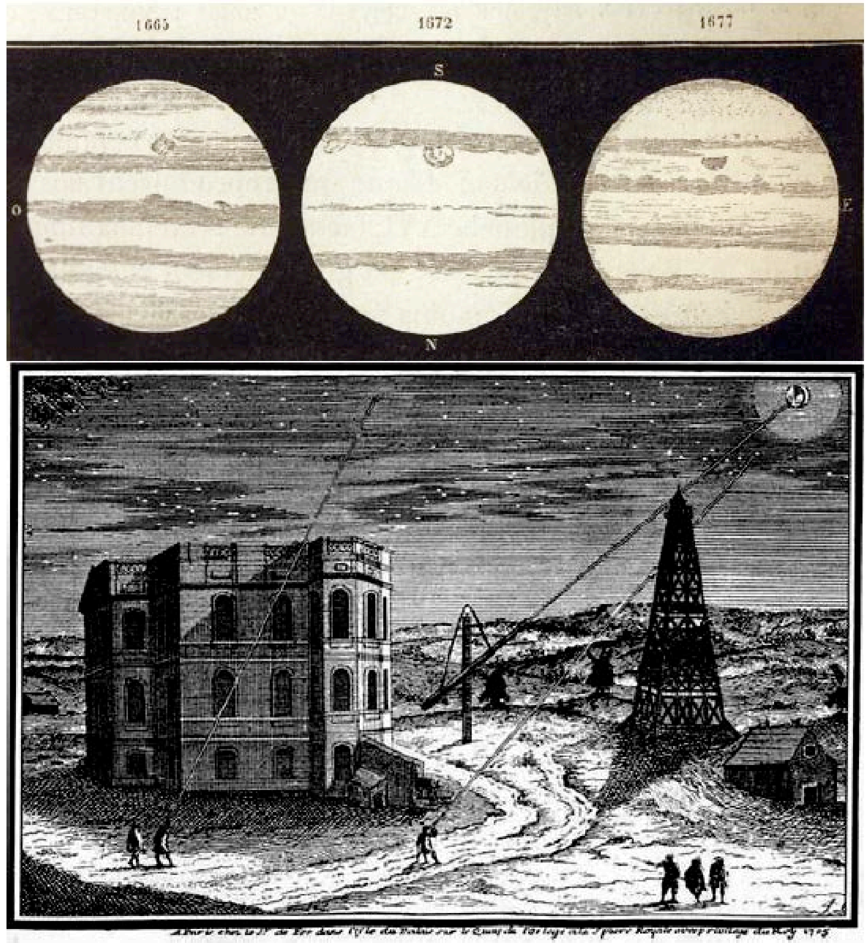
Frobenius-Perron – P - advance density,

Koopman - U - evaluate a function at the image,

$U \circ P[\rho_0](x) = U[\rho_1](x) = \rho_1 \circ T(x)$ evaluate the new density at the image point,

-or pull back density and look for almost invariant sets

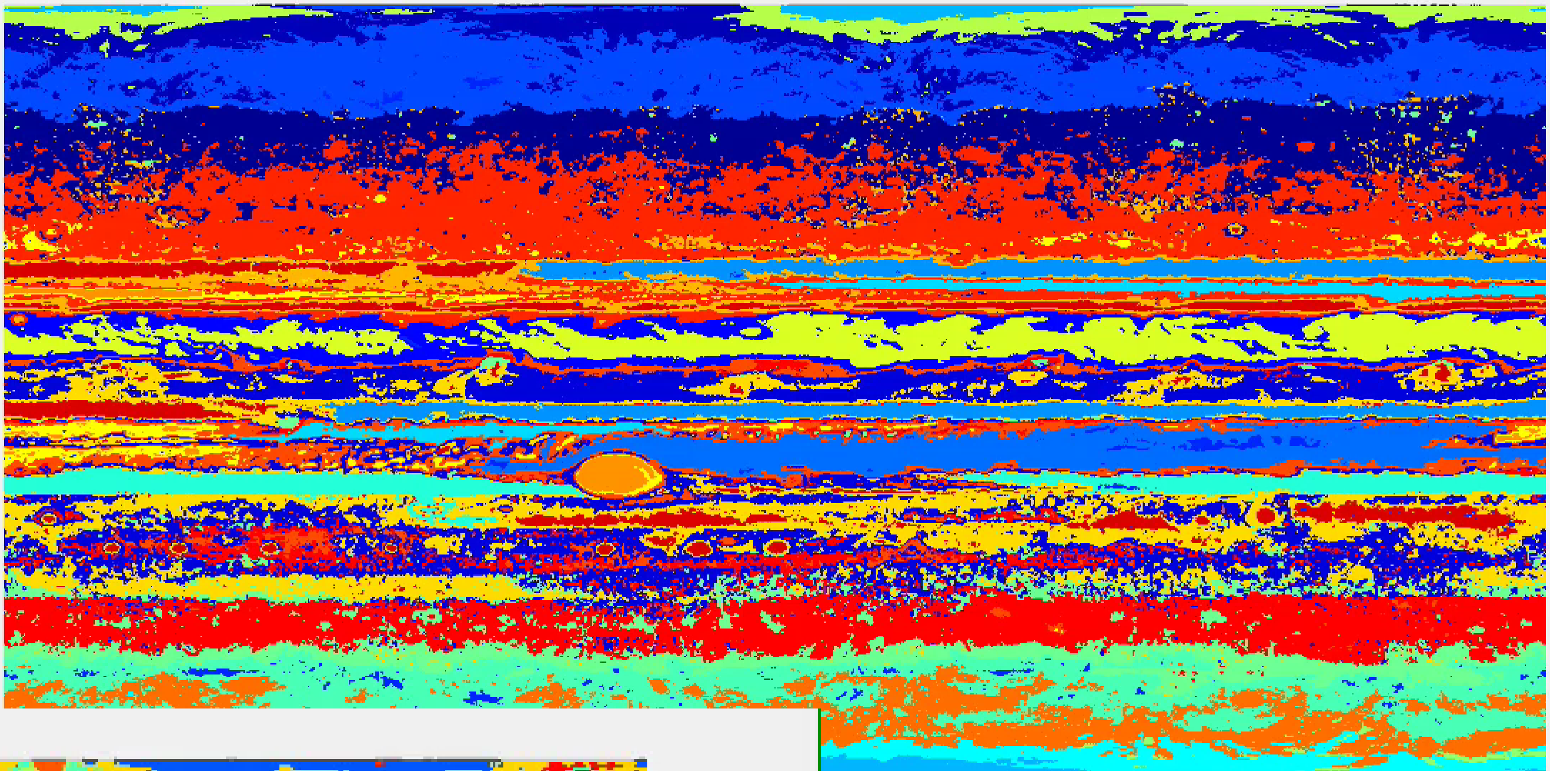
Can Coherence be Found without a Vector Field? Just based on what we watch?

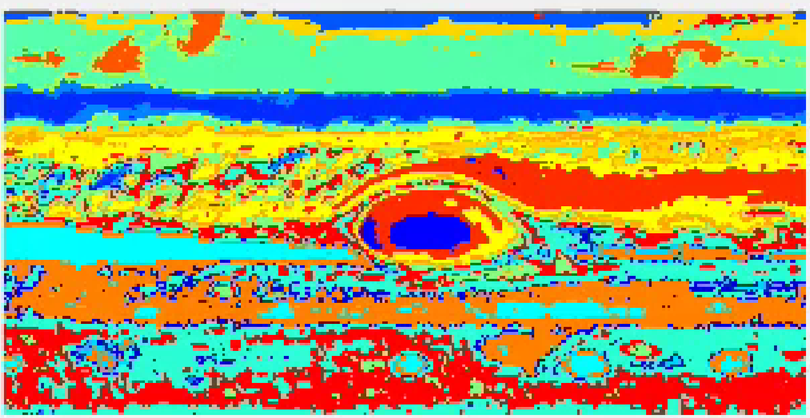


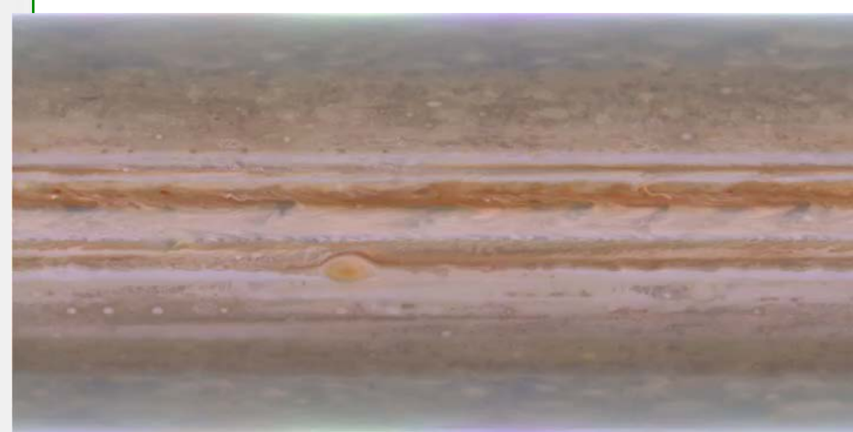
Jupiter as sketched by Giovanni Domenico Cassini (Top) in his own hand from 1665-1677, from the Memoires de l'Académie Royale des Sciences de Paris,[2]. Note that North is drawn, and so labelled, on the bottom. We see that Cassini was seeing and sketching similar scenes over the several years, including apparently the large storm. (Bottom) A sketch of the observatory in Paris.

Goal – coherent when we see it, and
to develop appropriate mathematical formalism behind this sense.

A punch-line first – Looking for a “Motion Segmentation” without a vector field







From The parallel worlds of spectral partitioning, graph theory, data analytics, image processing and dynamical systems coherency

Dhillon, I.S. and Guan, Y. and Kulis, B. (2004). Kernel k-means: spectral clustering and normalized cuts". Proceedings of the tenth ACM SIGKDD international conference on Knowledge discovery and data mining. pp. 551556.

Jianbo Shi and Jitendra Malik, Normalized Cuts and Image Segmentation," IEEE Transactions on Pattern Analysis and Machine Intelligence, (2000).

Y. Ng, M. I. Jordan, Y. Weiss: On spectral clustering: Analysis and an algorithm. In: 24 Advances in neural information processing systems. 2, (2002), S. 849-856.

Perona, W. T. Freeman, A Factorization Approach to Grouping," Proceedings of the 5th European Conference on Computer Vision Vol I 665-670 (1998).

Kannan, S. Vempala, A. Vetta, On clusterings - good, bad and spectral," Proceedings of the 41st Annual Symposium on Foundations of Computer Science, (2000).

Meila, M., Shi J., Learning segmentation by random walks," NIPS 2000.

Meila, M., Shi J., A random walks view of spectral segmentation," AISTATS 2001.

M. Meila, J. Shi, Learning segmentation by random walks," Neural Information Processing Systems, 13 (2001).

Dhillon, I.S. and Guan, Y. and Kulis, B. (2004). ``Kernel k-means: spectral clustering and normalized cuts". Proceedings of the tenth ACM SIGKDD international conference on Knowledge discovery and data mining. pp. 551-556.

M. Belkin and P. Niyogi. Laplacian eigenmaps and spectral techniques for embedding and clustering, NIPS Vol. 14, 2002.

Ronald R. Coifman, Stéphane Lafon. Diffusion maps. Appl. Comput. Harmon. Anal. 21 (2006) 5–30.

Kanungo, T., Mount, D. M., Netanyahu, N., Piatko, C., Silverman, R., and Wu,A. The efficient K-means clustering algorithm: analysis and implementation. IEEE Trans. PatternAnalysis Mach. Intell. 2002, 24(7), 881–892

- There have been several complementary views of clustering by spectral methods, by graph cuts as random walkers, and comparably as a diffusion process as described by diffusion map and comparably as an eigensystem.
- Weighted Directed Spectral Graph Theory has special issues related to the conductance of a graph (often called Cheeger-constant) which is numerical measure of whether or not a graph has a "bottleneck", and it given by:

$$\emptyset(G) = \min_{S \subseteq V} \varphi(S)$$

Where $\varphi(S) = \frac{\sum_{i \in S, j \in \bar{S}} a_{ij}}{\min(a(S), a(\bar{S}))}$ is the conductance of the cut $(S, \bar{S})$ of the graph G, and a_{ij} are entries of adjacency matrix of G.

$$\mathrm{X} = [\mathcal{C}_{1,:}^{\mathrm{T}} | \mathcal{C}_{2,:}^{\mathrm{T}} | .. | \mathcal{C}_{\mathrm{pq},:}^{\mathrm{T}}]$$

So for color alone, X is d×pq, and write $\mathrm{X_i} = \mathcal{C}_{i,:}^{\mathrm{T}}$ the column vector of colors at pixel position i. Then the pairwise distance is:

$$\mathrm{D_{i,j}} = \|\mathrm{X_i} - \mathrm{X_j}\|_2$$

Which gives the pairwise symmetric affinity matrix:

$$\mathrm{W_{i,j}} = e^{-\mathrm{D_{i,j}^2}/2\sigma^2}.$$

Degree matrix:

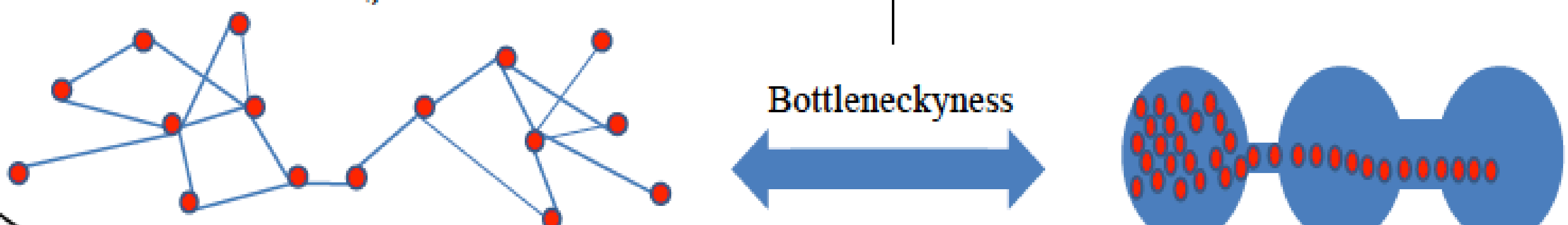
$$\mathcal{D}_{\mathrm{i,i}} = \sum_{\mathrm{j}} \mathrm{W_{i,j}}, \qquad \mathcal{D}_{\mathrm{i,j}} = 0, \qquad \mathrm{i} \neq \mathrm{j}$$

Then we have maximum cut problem:

$$\min_{\mathrm{x}} \mathrm{ncut(x)} = \min_{\mathrm{y}} \frac{\mathrm{y^T}(\mathcal{D} - \mathrm{W})\mathrm{y}}{\mathrm{y^T}\mathcal{D}\mathrm{y}}$$

Which solved by the generalized eigenvalue eigenvector problem

$$(\mathcal{D} - \mathrm{W})\mathrm{y} = \lambda \mathcal{D}\mathrm{y}$$



Weighted Directed Affinity For Developing "Motion Segmentation" interpretation of coherence without a vector field

$$X = \left[\mathcal{C}_{1,:}(t)^T \middle| \mathcal{C}_{2,:}(t)^T \middle| .. \middle| \mathcal{C}_{pq,:}(t)^T\right]$$

$D_1(i,j,a,\tau) = \sum_{l=1}^{\tau-1} \left\| X_i(t+la) - X_j(t+(l-1)a) \right\|_2$, Measure Affinity

$D_2(i,j)^2 = \|z(i) - z(j)\|_2^2$, Spatial Displacement Affinity

$$D(i,k,a,\tau)^2 = D_1(i,j,a,\tau)^2 + \alpha D_2(i,j)^2$$

The Weighted Directed Affinity

$$\mathcal{W}_{i,j} = e^{-D(i,k,a,\tau)^2/2\sigma^2}$$

<- reminds of diffusion map method?

$$\mathcal{P} = \mathcal{D}^{-1}\mathcal{W}$$

← This serves instead of the unknown
← transfer operator (as Naïve Bayes)

Where $\mathcal{D}$ is the degree matrix and $\mathcal{P}$ is a row stochastic matrix representing probabilities of a Markov chain through the directed graph.

Note: Affinity is based only in "affinity" from movie, no trajectories or transfer operators.

Directed Graph Analogue of Graph Laplacian

$$\mathcal{L} = I - \frac{\Pi^{1/2} P \Pi^{-1/2} + \Pi^{-1/2} P^T \Pi^{1/2}}{2}$$

Where Π is the diagonal matrix from the eigenvalue problem $u = u\mathcal{P}$, and $\Pi = \text{diag}(u)$.

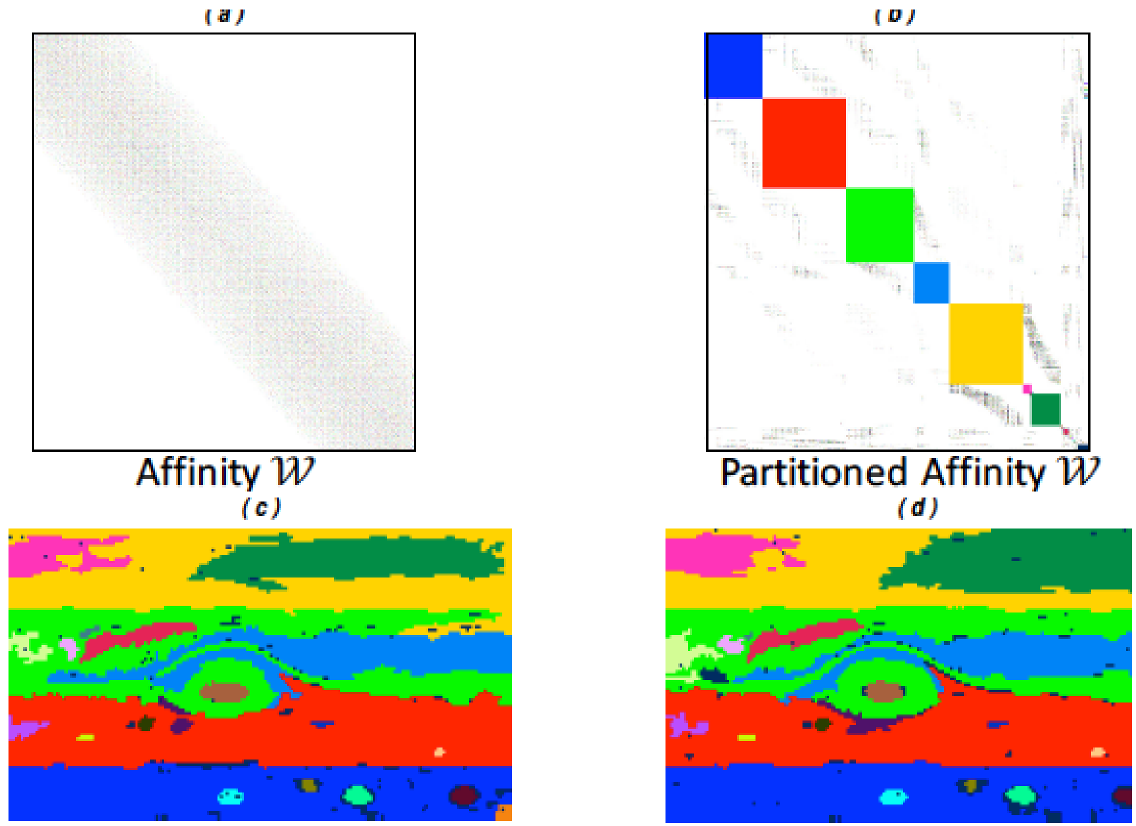
Numerical Results

Courant-Fischer Formula and Raleigh Quotient

Then eigenvectors cluster,
Max-cut problem for **directed**
– bottleneckyness/Cheeger/
conductance

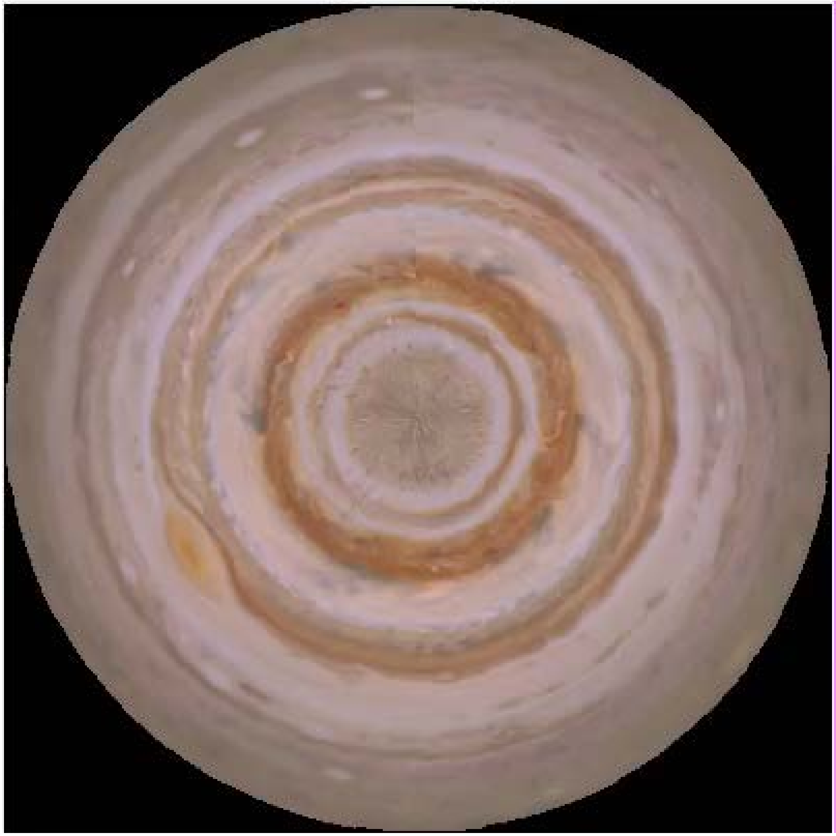
From Chung,
-already standard FP operators

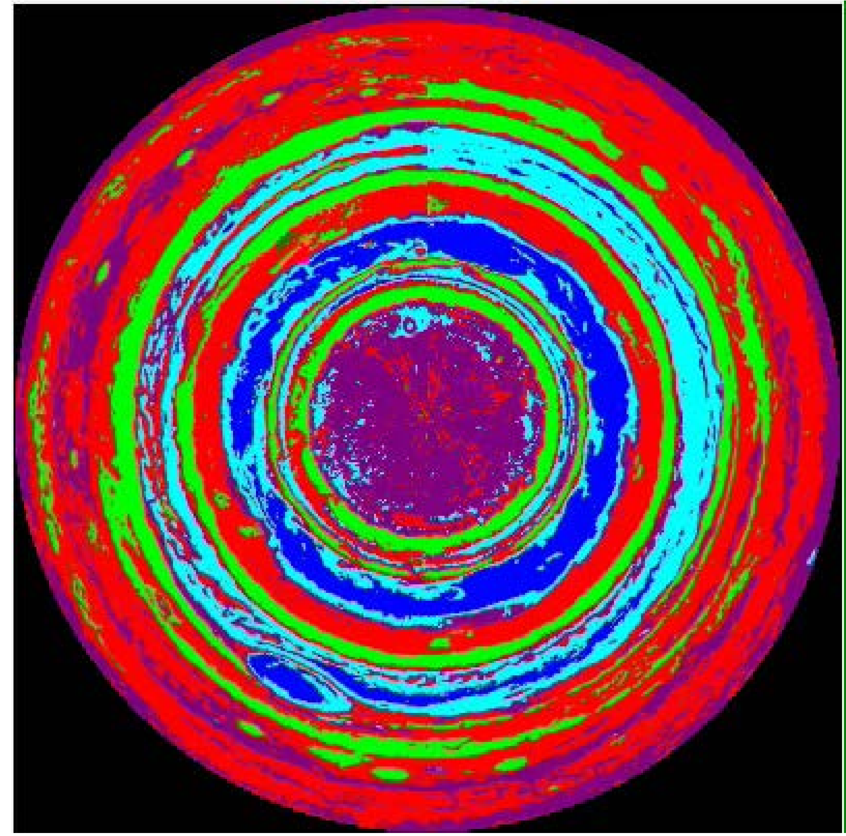
Fan Chung, Laplacians and the Cheeger inequality for directed graphs,"
Annals Combinatorics, 9 (2005), 1 19.

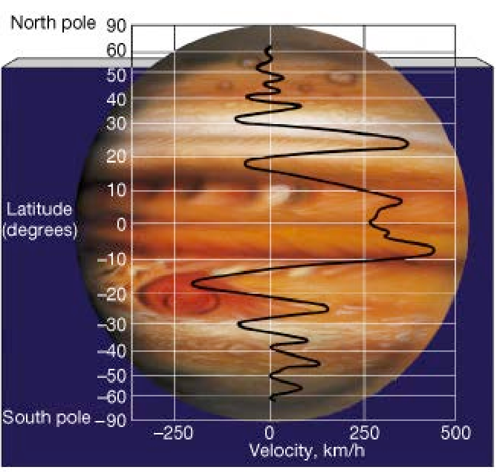


$$\lambda_2 = \min_{x^T p^{1/2}=0, x\neq 0} \frac{x^T \mathcal{L} x}{x^T x} = \min_{y^T p=0, y\neq 0} \frac{\sum_{i,j}(y_i - y_j)^2 p_i P_{i,j}}{\sum_i y_i^2 p_i}$$

Results – From Top View







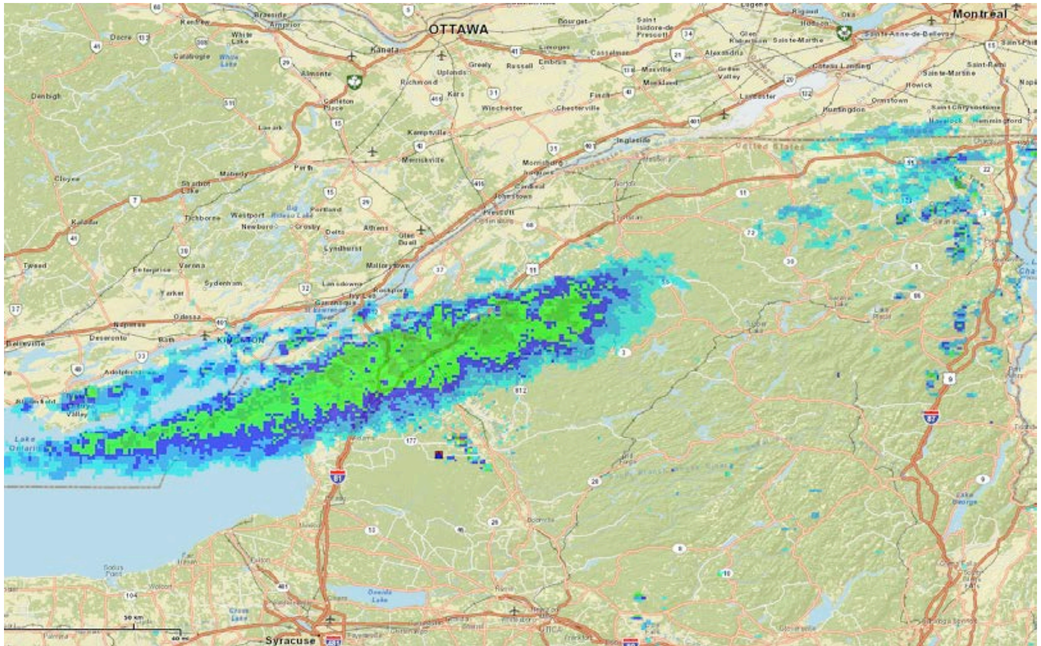
Zonal Flow The wind speed in Jupiter's atmosphere, measured relative to the planet's internal rotation rate. Alternations in wind direction are associated with the atmospheric band structure.

What is this?

- Coherent Sets?
- Clustered Time Evolving Measurements?
- Advection only?

Does observed coherence correspond to the same coherence one would get with A transfer operator from an advective process?

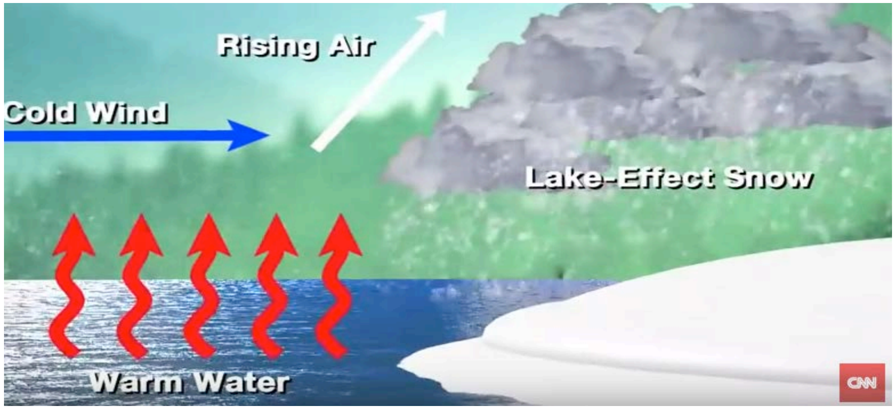
Weather- reaction (diffusive) advective, vs purely advective

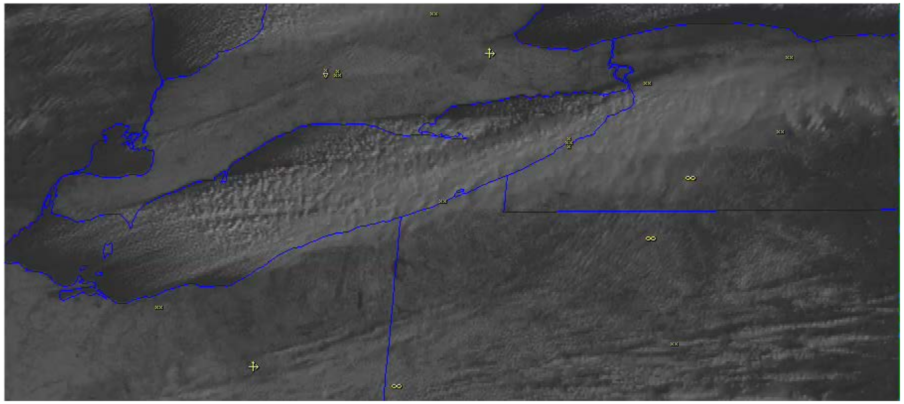


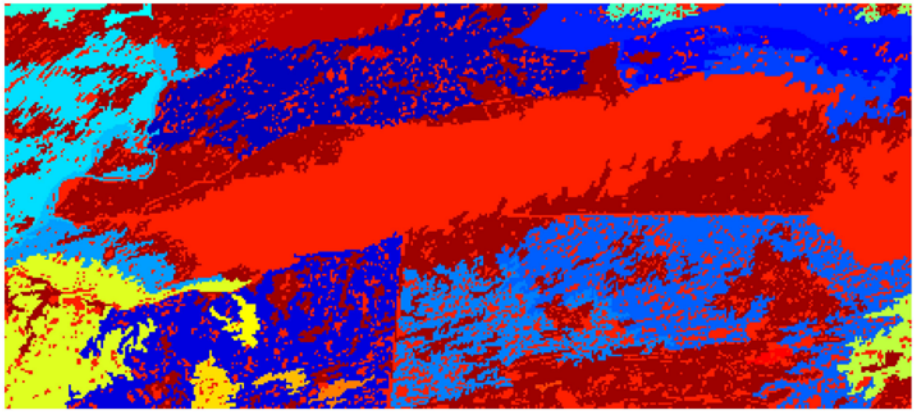
Cupid Note to Self:

Never Fall Asleep in a Blizzard

Weather- advective?



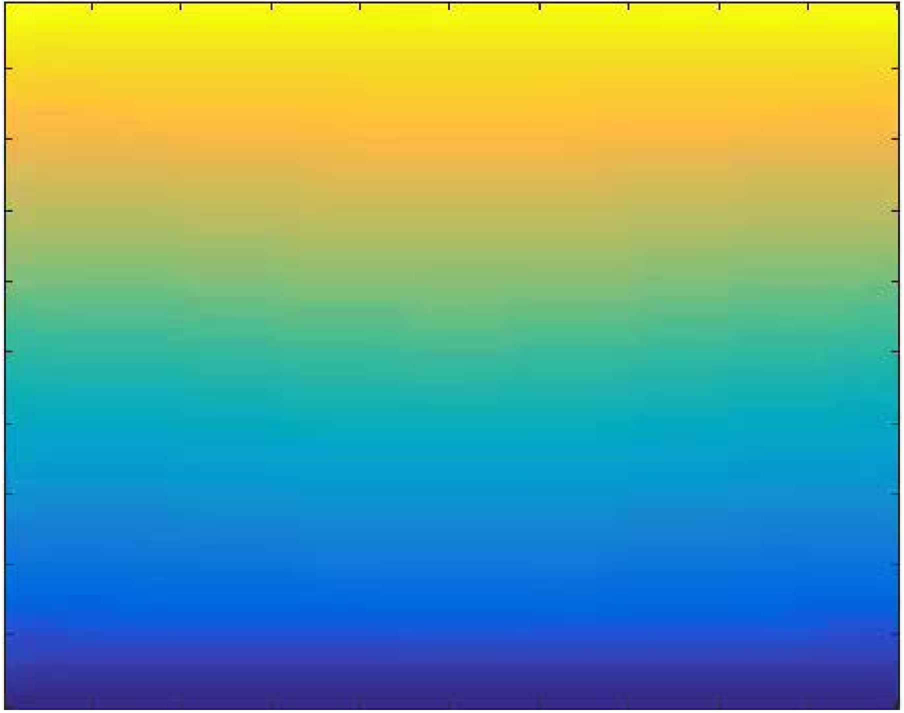


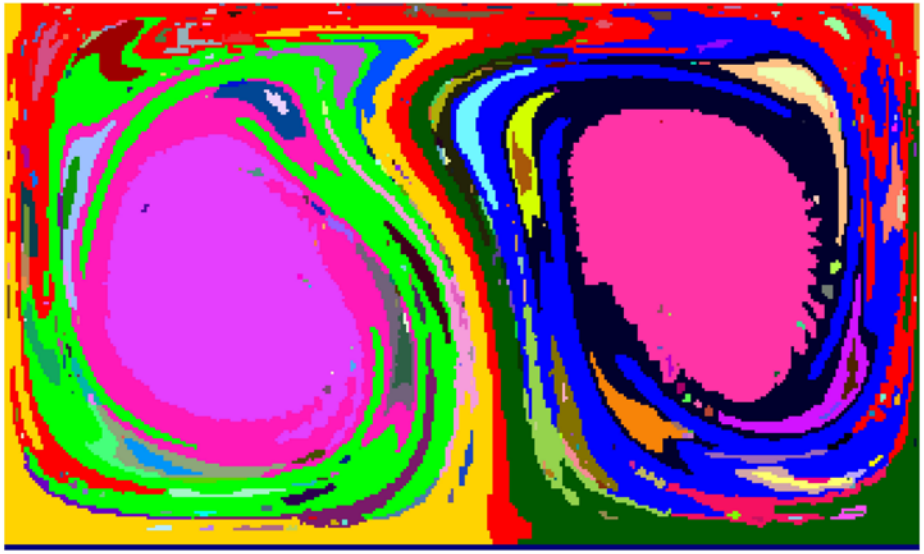


And all of our **favorite** benchmark test
In the subject of coherency

The “Harmonic Oscillator” of coherency

The **Double Gyre**





The End

-If you have the vector field, us it
-If you have the transfer operator, us it
-Coherence as a spectral motion tracking

-Modified with a weighted directed graph
    graph Laplacian – background of Cheeger
    by Fischer-Courant and Raleigh Quotient

-Details in papers on my website, and on the weighted directed graph Laplacian in my book
-Relate to an advective diffusion map